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MULTI-PARTICLE DYNAMICAL LOCALIZATION IN A CONTINUOUS ANDERSON MODEL WITH AN ALLOY-TYPE POTENTIAL

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Abstract. This paper is a complement to our earlier work [4]. With the help of the multi-scale analysis, we derive, from estimates obtained in [4], dynamical localization for a multi-particle Anderson model in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, with a short-range interaction, subject to a random alloy-type potential.

1. Introduction

1.1. **The model.** In this paper we continue our study of a multi-particle Anderson model in $\mathbb{R}^d$ with interaction and in an external random potential of alloy type. The Hamiltonian $\mathbf{H}$ $\big(= \mathbf{H}^{(N)}(\omega)\big)$ is a random Schrödinger operator of the form

$$\mathbf{H} = -\frac{1}{2}\mathbf{\Delta} + \mathbf{U}(\mathbf{x}) + \mathbf{V}(\omega;\mathbf{x}) \tag{1.1}$$

acting in $L^2(\mathbb{R}^{Nd})$. This means that we consider a system of N interacting quantum particles in $\mathbb{R}^d$. Here $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$ is for the joint position vector, where each component $x_j \in \mathbb{R}^d$ represents the position of the jth particle, $1 \leq j \leq N$. Next, $\mathbf{\Delta}$ stands for the Laplacian in $\mathbb{R}^{Nd}$. The interaction energy operator $\mathbf{U}(\mathbf{x})$ acts as multiplication by a function $U(\mathbf{x})$. Finally, the term $\mathbf{V}(\omega;\mathbf{x})$ represents the operator of multiplication by a function

$$\mathbf{x} \mapsto V(x_1;\omega) + \dots + V(x_N;\omega), \tag{1.2}$$

where $x \in \mathbb{R}^d \mapsto V(x;\omega)$ is a random external field potential assumed to be of the form

$$V(x;\omega) = \sum_{s\in\mathbb{Z}^d} \mathrm{V}_s(\omega)\,\varphi(x-s). \tag{1.3}$$

Here and below V_s, $s \in \mathbb{Z}^d$, are i.i.d. (independent and identically distributed) real random variables on some probability space $(\Omega, \mathfrak{B}, \mathbb{P})$ and $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is usually referred to as a "bump" function.

1.2. **Basic geometric notations.** Throughout this paper, we will fix an integer $N \geq 2$ and work in Euclidean spaces of the form $\mathbb{R}^{ld} \cong \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ (l times) associated with l-particle sub-systems where $1 \leq l \leq N$. Correspondingly, the notations $\mathbf{x}$, $\mathbf{y}, \ldots$ will be used for vectors from $\mathbb{R}^{ld}$, depending on the context. Given a vector $\mathbf{x} \in \mathbb{R}^{ld}$, we will consider "sub-configurations" $\mathbf{x}'$ and $\mathbf{x}''$ generated by $\mathbf{x}$ for a given partition of an l-particle system into disjoint sub-systems with l' and l'' particles, where $l' + l'' = l$, $l', l'' \geq 1$; the vectors $\mathbf{x}'$ and $\mathbf{x}''$ are identified with points from $\mathbb{R}^{l'd}$ and $\mathbb{R}^{l''d}$, respectively, by re-labelling the particles accordingly.

All Euclidean spaces will be endowed with the max-norm denoted by $|\cdot|$. We will consider ld-dimensional cubes of integer size in $\mathbb{R}^{ld}$ centered at lattice points $\mathbf{u} \in \mathbb{Z}^{ld} \subset \mathbb{R}^{ld}$ and with edges parallel to the co-ordinate axes. The cube of edge length $2L$ centered at $\mathbf{u}$ is denoted by $\boldsymbol{\Lambda}_L(\mathbf{u})$; in the max-norm it represents the ball of radius L centered at $\mathbf{u}$:

$$\boldsymbol{\Lambda}_L(\mathbf{u}) = \{\mathbf{x} \in \mathbb{R}^{ld} : \ |\mathbf{x} - \mathbf{u}| < L\}. \tag{1.4}$$

The lattice counterpart for $\boldsymbol{\Lambda}_L(\mathbf{u})$ is denoted by $\mathbf{B}_L(\mathbf{u})$:

$$\mathbf{B}_L(\mathbf{u}) = \overline{\boldsymbol{\Lambda}}_L(\mathbf{u}) \cap \mathbb{Z}^{ld}; \quad \mathbf{u} \in \mathbb{Z}^{ld}.$$

Finally, we consider "cells" (cubes of radius 1) centered at lattice points $\mathbf{u} \in \mathbb{Z}^{ld}$:

$$\mathbf{C}(\mathbf{u}) = \boldsymbol{\Lambda}_1(\mathbf{u}) \subset \mathbb{R}^{ld}.$$

The union of all cells $\mathbf{C}(\mathbf{u})$, $\mathbf{u} \in \mathbb{Z}^{ld}$, covers the entire Euclidean space $\mathbb{R}^{ld}$. For each $i \in \{1, \dots, l\}$ we introduce the projection $\Pi_i \colon \mathbb{R}^{ld} \to \mathbb{R}^d$ defined by

$$\Pi_i \colon (x_1, \dots, x_l) \longmapsto x_i, \ \ 1 \le i \le l.$$

1.3. **Interaction potential.** The interaction within the system of particles is represented by the term $\mathbf{U}(\mathbf{x})$ in the expression (1.1) of the Hamiltonian $\mathbf{H}$. As was said, it is the operator of multiplication by a function $\mathbf{x} \in \mathbb{R}^{ld} \mapsto U(\mathbf{x}) \in \mathbb{R}$, $1 \le l \le N$. A usual assumption is that $U(\mathbf{x})$ (considered for $\mathbf{x} \in \mathbb{R}^{ld}$ with $1 \le l \le N$) is a sum of k-body potentials

$$U(\mathbf{x}) = \sum_{k=1}^{l} \sum_{1 \le i_1 < \ldots < i_k \le l} U^{(k)}(x_{i_1}, \dots, x_{i_k}), \qquad \mathbf{x} = (x_1, \dots, x_l) \in \mathbb{R}^{ld}.$$

In this paper we do not assume isotropy, symmetry or translation invariance of this interaction. However, we use the conditions of finite range, nonnegativity and boundedness, as stated below.

Assume a partition of a configuration $\mathbf{x} \in \mathbb{Z}^{ld}$ is given, into complementary subconfigurations $\mathbf{x}_{\mathcal{J}} = (x_j)_{j \in \mathcal{J}}$ and $\mathbf{x}_{\mathcal{J}^c} = (x_j)_{j \in \{1,\dots,l\} \setminus \mathcal{J}}$, where $\varnothing \ne \mathcal{J} \subsetneq \{1, 2, \dots, l\}$. The *energy of interaction* between $\mathbf{x}_{\mathcal{J}}$ and $\mathbf{x}_{\mathcal{J}^c}$ is defined by

$$U(\mathbf{x}_{\mathcal{J}} \,|\, \mathbf{x}_{\mathcal{J}^c}) := U(\mathbf{x}) - U(\mathbf{x}_{\mathcal{J}}) - U(\mathbf{x}_{\mathcal{J}^c}). \tag{1.5}$$

Next, define

$$\rho(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}) := \min \ \Big[|x_i - x_j| : \ i \in \mathcal{J}, j \in \mathcal{J}^{\mathrm{c}} \Big]. \tag{1.6}$$

We say that this interaction has *range* $\mathrm{r}_0 \in (0, \infty)$ if, for all $l = 1, \dots, N$ and $\mathbf{x} \in \mathbb{R}^{ld}$,

$$\rho(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}) > \mathrm{r}_0 \implies U(\mathbf{x}_{\mathcal{J}} \,|\, \mathbf{x}_{\mathcal{J}^c}) = 0. \tag{1.7}$$

Finally, we say that the interaction is *non-negative* and *bounded* if

$$\inf_{\mathbf{x} \in \mathbb{R}^{ld}} U(\mathbf{x}) \ge 0 \ \text{ and } \ \sup_{\mathbf{x} \in \mathbb{R}^{ld}} U(\mathbf{x}) < +\infty, \quad 1 \le l \le N. \tag{1.8}$$

The boundedness condition can be relaxed to include hard-core interactions where $U(\mathbf{x}) = +\infty$ if $|x_i - x_j| \le a$, for some given $a \in (0, \mathrm{r}_0)$.

1.4. **Assumptions.** Our assumptions on the interaction potential U are borrowed from [4]:

(E1) U is non-negative, bounded and has a finite range $\mathrm{r}_0 \geq 0$.

Similarly, we use assumptions on the i.i.d. random variables V_s, $s \in \mathbb{Z}^d$, and the bump function φ introduced in [4]:

(E2) There exists a constant $\mathrm{v} \in (0, \infty)$ such that

$$\mathbb{P}\{0 \leq \mathrm{V}_0 \leq \mathrm{v}\} = 1 \tag{1.9}$$

and

$$\forall\, \epsilon > 0 \quad \mathbb{P}\{\mathrm{V}_0 \leq \epsilon\} > 0. \tag{1.10}$$

(E3) *Uniform Hölder continuity*:[1] There exist constants a, b > 0 such that for all $\epsilon \in [0, 1]$, the common distribution function F of the random variables V_s satisfies

$$\sup_{\mathrm{y} \in \mathbb{R}} \big[F(\mathrm{y} + \epsilon) - F(\mathrm{y})\big] \leq \mathrm{a}\epsilon^{\mathrm{b}}. \tag{1.11}$$

(E4) The function $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ is bounded, nonnegative and compactly supported:

$$\operatorname{diam}(\operatorname{supp} \varphi) \leq \mathrm{r}_1 < \infty. \tag{1.12}$$

(E5) For all $L \geq 1$ and $u \in \mathbb{Z}^d$,

$$\sum_{s \in \Lambda_L(u) \cap \mathbb{Z}^d} \varphi(x - s) \geq \mathbf{1}_{\Lambda_L(u)}(x). \tag{1.13}$$

Here and below, $\mathbf{1}_A$ stands for the indicator function of a set A.

Henceforth, we suppose that d and N are fixed, as well as the interaction $\mathbf{U}$ and the structure of the external potential (i.e., the distribution function F and the bump function φ). All constants emerging in various bounds below are introduced under this assumption.

1.5. **Dynamical localization.** The main result of this paper, Theorem 1.1, establishes the so-called "strong dynamical localization" for the operator $\mathbf{H}(\omega)$ defined in (1.1) near the lower edge E^0 of its spectrum. More precisely, let E^0 be the lower edge of the spectrum $\operatorname{spec}(\mathbf{H}^0)$ of the N-particle operator without interaction,

$$\mathbf{H}^0 = -\frac{1}{2}\mathbf{\Delta} + \sum_{j=1}^{N} V(x_j; \omega). \tag{1.14}$$

Actually, it follows from our conditions (1.9) and (1.10) that $E^0 = 0$. Owing to the non-negativity of the interaction potential U, the lower edge of the spectrum of $\mathbf{H}$ is bounded from below by E^0. Moreover, $\mathbf{H}$ has a non-empty spectrum in the interval $[E^0, E^0 + \epsilon]$, for any $\epsilon > 0$. This follows, e.g., from a result by Klopp and Zenk [8] which says that the integrated density of states for a multi-particle system with a decaying interaction is the same as for the system without interaction.

Denote by $\mathbf{X}$ the operator of multiplication by the norm of $\mathbf{x}$, i.e.,

$$\mathbf{X} f(\mathbf{x}) = |\mathbf{x}|\, f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{Nd}. \tag{1.15}$$

The main result of this paper is the following

[1] The Hölder continuity can be relaxed to the log-Hölder continuity.

Theorem 1.1. *Consider the operator* $\mathbf{H}$ *from* (1.1) *and assume that conditions* **(E1)**–**(E5)** *are fulfilled. Then for any* $Q > 0$ *there exists a nonrandom number* $\eta = \eta(Q) > 0$ *such that for any compact subset* $\mathbf{K} \subset \mathbb{R}^{Nd}$ *the following bound holds:*

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| \mathbf{X}^Q \, \mathrm{e}^{-\mathrm{i}t\mathbf{H}(\omega)} P_{I(\eta)}(\mathbf{H}(\omega))\, \mathbf{1}_{\mathbf{K}} \right\|_{L^2(\mathbb{R}^{Nd})}\right] < \infty, \tag{1.16}$$

where $P_{I(\eta)}(\mathbf{H})$ *is the spectral projection of the Hamiltonian* $\mathbf{H}$ *on the interval* $I(\eta) = [E^0,\, E^0 + \eta]$.

Remark 1.2. The interval $I(\eta)$ is a sub-interval of the interval of energies $[E^0, E^0 + \eta^*]$ for which the spectrum of $\mathbf{H}$ was proven to be pure point (and the eigenfunctions to be decaying exponentially); see [4].

2. Results of the multi-particle MSA

The MSA works with the finite-volume approximations $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{u})}$ of $\mathbf{H}$, relative to the cubes $\mathbf{\Lambda}_L(\mathbf{u})$. More precisely, $\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{u})}$ is an operator in $L^2(\mathbf{\Lambda}_L(\mathbf{u}))$, given by the same expression as in (1.1) (for $\mathbf{x} \in \mathbf{\Lambda}_L(\mathbf{u})$), with Dirichlet's boundary conditions on $\partial\mathbf{\Lambda}_L(\mathbf{u})$; see [4]. Specifically, the Green operator $\mathbf{G}_{\mathbf{\Lambda}_L(\mathbf{u})}(E)$ is of particular interest:

$$\mathbf{G}_{\boldsymbol{\Lambda}_L(\mathbf{u})}(E) = (\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{u})} - E)^{-1}, \tag{2.1}$$

defined for $E \in \mathbb{R} \setminus \operatorname{spec}\big(\mathbf{H}_{\mathbf{\Lambda}_L(\mathbf{u})}\big)$.

Let $[\,\cdot\,]$ denote the integer part. For a cube $\boldsymbol{\Lambda}_L(\mathbf{u})$ we denote

$$\boldsymbol{\Lambda}^{\mathrm{int}}_L(\mathbf{u}) = \boldsymbol{\Lambda}_{[L/3]}(\mathbf{u}), \quad \boldsymbol{\Lambda}^{\mathrm{out}}_L(\mathbf{u}) = \boldsymbol{\Lambda}_L(u) \setminus \boldsymbol{\Lambda}_{L-2}(u). \tag{2.2}$$

Next, given two points $\mathbf{v}, \mathbf{w} \in \mathbf{B}_L(\mathbf{u})$ such that $\mathbf{C}(\mathbf{v}), \mathbf{C}(\mathbf{w}) \subset \boldsymbol{\Lambda}_L(\mathbf{u})$, set

$$\mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}_{\mathbf{v},\mathbf{w}}(E) := \mathbf{1}_{\mathbf{C}(\mathbf{v})}\, \mathbf{G}_{\boldsymbol{\Lambda}_L(\mathbf{u})}(E)\, \mathbf{1}_{\mathbf{C}(\mathbf{w})}\,. \tag{2.3}$$

Following a long-standing tradition, we use a parameter $\alpha \in (1, 2)$ in the definition of a sequence of scales L_k (cf. Eqn (2.5)); For our purposes, it suffices to set $\alpha = 3/2$; this will be always assumed below.

Definition 2.1. A cube $\boldsymbol{\Lambda}_L(\mathbf{u})$ is called (E, m)-*non-singular* ((E, m)-NS, in short) if for any $\mathbf{v} \in \mathbf{B}_{\left[L^{1/\alpha}\right]}(\mathbf{u})$ and $\mathbf{y} \in \boldsymbol{\Lambda}^{\mathrm{out}}_L(\mathbf{u}) \cap \mathbb{Z}^{Nd}$ the norm of the operator $\mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}_{\mathbf{v},\mathbf{y}}(E)$ satisfies

$$\left\| \mathbf{G}^{\boldsymbol{\Lambda}_L(\mathbf{u})}_{\mathbf{v},\mathbf{y}}(E) \right\|_{L^2(\mathbf{\Lambda}_L(\mathbf{u}))} \le \mathrm{e}^{-mL}. \tag{2.4}$$

Otherwise, it is called (E, m)-*singular* ((E, m)-S).

We will work with a sequence of "scales" L_k (positive integers) defined recursively by

$$L_k := \left[L^{\alpha}_{k-1}\right] + 1, \ \text{ where } \ \alpha = \frac{3}{2}\,. \tag{2.5}$$

The sequence L_k is determined by an initial scale $L_0 \ge 2$. Most of arguments in Sect. 3 require L_0 to be large enough, to fulfill some specific numerical inequalities. In addition, we also assume that $L_0 \ge \mathrm{r}_1$ (defined in (1.12)) in order to simplify some cumbersome technicalities.

We will use a well-known property of generalized eigenfunctions of the operator $\mathbf{H}$ which can be found, e.g., in [9, Lemma 3.3.2]:

Lemma 2.2. *For every bounded set $I_0 \subset \mathbb{R}$ there exists a constant $C^{(0)} = C^{(0)}(I_0)$ such that, for any cube $\mathbf{\Lambda}_L(\mathbf{u})$ with $L > 7$, any point $\mathbf{w} \in \mathbf{B}_L(\mathbf{u})$ with $\mathbf{C}(\mathbf{w}) \subseteq \mathbf{\Lambda}_L^{\rm int}(\mathbf{u})$ and every generalized eigenfunction Ψ of $\mathbf{H}$ with eigenvalue $E \in I_0$, the norm of the vector $\mathbf{1}_{\mathbf{C}(\mathbf{w})} \boldsymbol{\Psi}$ satisfies*

$$\| \mathbf{1}_{\mathbf{C}(\mathbf{w})} \boldsymbol{\Psi} \| \le C^{(0)} \, \| \mathbf{1}_{\mathbf{\Lambda}_L^{\rm out}(u)} \mathbf{G}_{\boldsymbol{\Lambda}_L(\mathbf{u})}(E) \; \mathbf{1}_{\mathbf{C}(\mathbf{w})} \| \cdot \| \mathbf{1}_{\mathbf{\Lambda}_L^{\rm out}(\mathbf{u})} \boldsymbol{\Psi} \|. \tag{2.6}$$

(From now on we omit the subscript indicating the L^2-space where a given norm is considered, as this will be clear in the context of the argument.)

The following geometric notion is used in the forthcoming analysis.

Definition 2.3. (see [4]). Let $\mathcal{J}$ be a non-empty subset of $\{1, \dots, N\}$.

We say that the cube $\boldsymbol{\Lambda}_L(\mathbf{y})$ is *$\mathcal{J}$-separable* from the cube $\boldsymbol{\Lambda}_L(\mathbf{x})$ if

$$\left(\bigcup_{j \in \mathcal{J}} \Pi_j \boldsymbol{\Lambda}_{L+\mathrm{r}_1}(\mathbf{y}) \right) \bigcap \left(\bigcup_{i \notin \mathcal{J}} \Pi_i \boldsymbol{\Lambda}_{L+\mathrm{r}_1}(\mathbf{y}) \; \bigcup \Pi \boldsymbol{\Lambda}_{L+\mathrm{r}_1}(\mathbf{x}) \right) = \varnothing \tag{2.7}$$

where $\Pi \boldsymbol{\Lambda}_{L+\mathrm{r}_1}(\mathbf{x}) = \cup_{j=1}^N \Pi_j \boldsymbol{\Lambda}_{L+\mathrm{r}_1}(\mathbf{x})$.

A pair of cubes $\boldsymbol{\Lambda}_L(\mathbf{x})$, $\boldsymbol{\Lambda}_L(\mathbf{y})$ is *separable* if, for some $\mathcal{J} \subseteq \{1, \dots, N\}$, either $\boldsymbol{\Lambda}_L(\mathbf{y})$ is $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L(\mathbf{x})$, or $\boldsymbol{\Lambda}_L(\mathbf{x})$ is $\mathcal{J}$-separable from $\boldsymbol{\Lambda}_L(\mathbf{y})$.

We will use the following easy assertion (see [4]):

Lemma 2.4. *For any $L > 1$ and $\mathbf{x} \in \mathbb{R}^{Nd}$, there exists a collection of N-particle cubes $\boldsymbol{\Lambda}_{2N(L+\mathrm{r}_1)}(\mathbf{x}^{(l)})$, $l = 1, \dots, K(\mathbf{x}, N)$, with $K(\mathbf{x}, N) \le N^N$, such that if a vector $\mathbf{y} \in \mathbb{Z}^{Nd}$ satisfies*[2]

$$\mathbf{y} \notin \bigcup_{\ell=1}^{K(\mathbf{x},N)} \boldsymbol{\Lambda}_{2N(L+\mathrm{r}_1)}(\mathbf{x}^{(l)}), \tag{2.8}$$

then two cubes $\boldsymbol{\Lambda}_L(\mathbf{x})$ and $\boldsymbol{\Lambda}_L(\mathbf{y})$ with $\operatorname{dist}(\boldsymbol{\Lambda}_L(\mathbf{x}), \boldsymbol{\Lambda}_L(\mathbf{y})) > 2N(L+\mathrm{r}_1)$ *are separable. In particular, assuming $L \ge \mathrm{r}_1$, a pair of cubes $\boldsymbol{\Lambda}_L(\mathbf{x})$, $\boldsymbol{\Lambda}_L(\mathbf{y})$ is separable if*

$$|\mathbf{y}| > |\mathbf{x}| + (4N+2)L. \tag{2.9}$$

Since $N \ge 2$, one can replace the condition (2.9) by

$$|\mathbf{y}| > |\mathbf{x}| + 5NL. \tag{2.10}$$

In particular, two cubes of the form $\boldsymbol{\Lambda}_L(\mathbf{0})$, $\boldsymbol{\Lambda}_L(\mathbf{y})$ with $|\mathbf{y}| > 5NL$ are always separable.

The main outcome of [4] is summarized in the following Theorem 2.5:

Theorem 2.5 (see [4])**.** *For any large enough $p > 0$ there exist $m^*(p) > 0$, $\eta^*(p) > 0$, and $L_0^*(p) > 0$ such that*

(i) *if $L_0 \ge L_0^*(p)$ then for all $k \ge 0$ and for any pair of separable cubes $\boldsymbol{\Lambda}_{L_k}(\mathbf{x})$, $\boldsymbol{\Lambda}_{L_k}(\mathbf{y})$ with $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$,*

$$\mathbb{P}\left\{ \exists \, E \in [E^0, E^0 + \eta^*] : \; \boldsymbol{\Lambda}_{L_k}(\mathbf{x}) \text{ and } \boldsymbol{\Lambda}_{L_k}(\mathbf{y}) \text{ are } (E,m)\text{-S} \right\} \le L_k^{-2p}, \tag{2.11}$$

(ii) *with probability one, the spectrum of $\mathbf{H}$ in the interval $I = [E^0, E^0 + \eta^*(p)]$ is pure point, and the eigenfunctions $\boldsymbol{\Phi}_n$ of $\mathbf{H}$ with eigenvalues $E_n \in I$ satisfy*

$$\|\boldsymbol{\Phi}_n \, \mathbf{1}_{\mathbf{C}(\mathbf{w})} \| \le C_n(\omega) \mathrm{e}^{-m^*(p)|\mathbf{w}|}, \quad \mathbf{w} \in \mathbb{Z}^{Nd}, \quad C_n(\omega) < \infty. \tag{2.12}$$

[2] The constant r_1 is defined in (1.12).

3. Derivation of dynamical localization from MSA estimates

In this section we prove a statement that is slightly more general than Theorem 1.1. Namely, given $Q > 0$, the interval $I = I(\eta) = [E^0, E^0 + \eta]$ with $\eta = \eta(Q)$, and a compact subset $\mathbf{K} \subset \mathbb{R}^d$, there exists a constant $C(Q, \mathbf{K}) \in (0, \infty)$ such that for any bounded measurable function $\xi \colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} \xi \subset I(\eta)$,

$$\mathbb{E}\left[\|\mathbf{X}^Q \, \xi(\mathbf{H}(\omega)) \, \mathbf{1}_{\mathbf{K}} \, \|\right] < C(Q, \mathbf{K}) \, \|\xi\|_\infty < \infty. \tag{3.1}$$

Moreover, $Q > 0$ can be made arbitrarily large, by choosing $\eta = \eta(Q) > 0$ sufficiently small. Theorem 1.1 follows from (3.1) applied to the functions $\xi(s) = \mathrm{e}^{-\mathrm{i}ts} \, \mathbf{1}_{I(\eta)}(s)$, parametrised by $t \in \mathbb{R}$.

Throughout the section, we assume that the parameter p from (2.11) satisfies

$$2p > 3Nd\alpha + \alpha Q. \tag{3.2}$$

More precisely, given $Q > 0$ and p satisfying (3.2), we work with

$$\eta = \eta(Q) \in (0, \eta^*(p)) \; \text{ and } \; m = m^*(p) > 0, \tag{3.3}$$

where $\eta^*(p)$ and $m^*(p)$ are specified in Theorem 2.5. Further, for p satisfying (3.2) we introduce the event $\Omega_1 = \Omega_1(p) \subseteq \Omega$ of probability $\mathbb{P}(\Omega_1) = 1$, defined by

$$\Omega_1 = \left\{\omega \in \Omega : \text{ the spectrum of } \mathbf{H}(\omega) \text{ in } [E^0, E^0 + \eta^*(p)] \text{ is pure point}\right\}. \tag{3.4}$$

3.1. Probability of "bad samples".

Given $j \geq 1$, consider the event

$$\begin{aligned} \mathcal{S}_j = \{\omega : \text{ there exists } E \in I \text{ and } \mathbf{y}, \mathbf{z} \in \mathbf{B}_{5NL_{j+1}}(\mathbf{0}) \text{ such that} \\ \mathbf{\Lambda}_{L_j}(\mathbf{y}), \mathbf{\Lambda}_{L_j}(\mathbf{z}) \text{ are separable and } (m, E)\text{-S}\}. \end{aligned}$$

Further, for $k \geq 1$ we denote

$$\Omega_k^{\mathrm{bad}} = \bigcup_{j \geq k} \mathcal{S}_j. \tag{3.5}$$

Lemma 3.1. *There exists a constant $c_1 \in (0, \infty)$ such that for all $k \geq 1$,*

$$\mathbb{P}\left\{\Omega_k^{\mathrm{bad}}\right\} \leq c_1 L_k^{-(2p - 2Nd\alpha)}.$$

Proof. The number of separable pairs $\mathbf{\Lambda}_{L_j}(\mathbf{x})$, $\mathbf{\Lambda}_{L_j}(\mathbf{y})$ with $\mathbf{x}, \mathbf{y} \in \mathbf{B}_{5NL_{j+1}}(\mathbf{0})$ is bounded by $(10NL_{j+1} + 1)^2 < C(N) L_{j+1}^2$. We can apply the bound (2.11) and write

$$\mathbb{P}\left\{\mathcal{S}_j\right\} \leq C(N) L_{j+1}^{2Nd} L_j^{-2p} \leq L_j^{-2p + 2Nd\alpha}.$$

Therefore,

$$\Omega_k^{\mathrm{bad}} \leq \; L_k^{-2p + 2Nd\alpha} \sum_{i \geq 0} \left(\frac{L_{k+i}}{L_k}\right)^{-2p + 2Nd\alpha}.$$

For $2p > 2Nd\alpha$ and $L_0 \geq 2$ the claim follows from the inequality

$$\frac{L_{k+i}}{L_k} \geq \left[L_k^{\alpha^i - 1}\right]. \qquad \square$$

3.2. **Centers of localization.** Denote by $\boldsymbol{\Phi}_n = \boldsymbol{\Phi}_n(\omega)$ the normalized eigenfunctions of $\mathbf{H}(\omega)$, $\omega \in \Omega_1$, with corresponding eigenvalues $E_n = E_n(\omega) \in I$. For each n we define a *center of localization* for $\boldsymbol{\Phi}_n$ as a point $\hat{\mathbf{x}} \in \mathbb{Z}^d$ such that

$$\| \mathbf{1}_{\mathbf{C}(\hat{\mathbf{x}})} \, \boldsymbol{\Phi}_n \| = \max_{\mathbf{y} \in \mathbb{Z}^{Nd}} \| \mathbf{1}_{\mathbf{C}(\mathbf{y})} \, \boldsymbol{\Phi}_n \|. \tag{3.6}$$

Since $\|\boldsymbol{\Phi}_n\| = 1$, for any given n such centers exist and their number is finite. We will assume that, for any eigenfunction $\boldsymbol{\Phi}_n$, the centers of localization $\hat{\mathbf{x}}_{n,a}$, $a = 1, \dots, \hat{C}(n)$, are enumerated in such a way that $|\hat{\mathbf{x}}_{n,1}| = \min_a |\hat{\mathbf{x}}_{n,a}|$.

Lemma 3.2. *There exists k_0 large enough such that, for all $\mathbf{u} \in \mathbb{Z}^{Nd}$, $\omega \in \Omega_1$ and $k \geq k_0$, if $\hat{\mathbf{x}}_{n,a} \in \mathbf{B}_{L_k}(\mathbf{u})$ then the box $\boldsymbol{\Lambda}_{L_k}(\mathbf{u})$ is (m, E_n)-S.*

Proof. Assume otherwise. Then from (2.6) it follows that

$$\| \mathbf{1}_{\mathbf{C}(\hat{\mathbf{x}}_{n,a})} \, \boldsymbol{\Phi}_n \| \leq C' \mathrm{e}^{-mL_k} \| \mathbf{1}_{\boldsymbol{\Lambda}^{\mathrm{out}}_{L_k}(\mathbf{u})} \, \boldsymbol{\Phi}_n \|.$$

Since the number of cells in $\boldsymbol{\Lambda}^{\mathrm{out}}_{L_k}(\mathbf{u})$ is bounded by L_k^{Nd}, we conclude that

$$\| \mathbf{1}_{\mathbf{C}(\hat{\mathbf{x}}_{n,a})} \, \boldsymbol{\Phi}_n \| \leq C' \mathrm{e}^{-mL_k} L_k^{Nd} \cdot \max_{\mathbf{y} \in \mathbf{B}^{\mathrm{out}}_{L_k}(\mathbf{u})} \| \mathbf{1}_{\mathbf{C}(\mathbf{y})} \, \boldsymbol{\Phi}_n \|.$$

If k_0 is large enough so that $C' \mathrm{e}^{-mL_k} L_k^{Nd} < 1$ for $k \geq k_0$, the above inequality contradicts the definition of $\hat{\mathbf{x}}_{n,a}$ as center of localization. ☐

3.3. **Annular regions.** From now on we work with the integer k_0 from Lemma 3.2. Given $k > k_0$, set:

$$\Omega_k^{\mathrm{good}} = \Omega_1 \setminus \Omega_k^{\mathrm{bad}}. \tag{3.7}$$

Assume that $\omega \in \Omega_k^{\mathrm{good}}$. Let $\hat{\mathbf{x}}_{n,a}$, $\hat{\mathbf{x}}_{n,b}$ be two centers of localization for the same eigenfunction $\boldsymbol{\Phi}_n$. It follows from the definition of the event Ω_k^{good} that the cubes $\boldsymbol{\Lambda}_{L_i}(\hat{\mathbf{x}}_{n,a})$ and $\boldsymbol{\Lambda}_{L_i}(\hat{\mathbf{x}}_{n,b})$ with $i \geq k-1$ cannot be separable, since they must be (m, E)-S. Further, by Lemma 2.4, if $L_0 \geq \mathrm{r}_1$ then any cube of the form $\boldsymbol{\Lambda}_{L_k}(\mathbf{y})$ with $|\mathbf{y}| > |\hat{\mathbf{x}}_{n,1}| + 5NL_k$ is separable from $\boldsymbol{\Lambda}_{L_k}(\hat{\mathbf{x}}_{n,1})$; this also applies, of course, to any localization center $\mathbf{y} = \hat{\mathbf{x}}_{n,a}$ with $a > 1$, provided that such centers exist for a given n. Since $\omega \in \Omega_k^{\mathrm{good}}$, for any eigenfunction $\boldsymbol{\Phi}_n$ there is no center of localization $\hat{\mathbf{x}}_{n,a}$ either outside the cube $\boldsymbol{\Lambda}_{|\hat{\mathbf{x}}_{n,1}|+5NL_k}(\mathbf{0})$ or inside $\boldsymbol{\Lambda}_{|\hat{\mathbf{x}}_{n,1}|}(\mathbf{0})$ (since $|\hat{\mathbf{x}}_{n,1}| = \min_a |\hat{\mathbf{x}}_{n,a}|$). In other words, within the event Ω_k^{good}, all centers of localization $\hat{\mathbf{x}}_{n,a}$ with a fixed value of n are located in the annulus

$$\boldsymbol{\Lambda}_{|\hat{\mathbf{x}}_{n,1}|+5NL_k}(\mathbf{0}) \setminus \boldsymbol{\Lambda}_{|\hat{\mathbf{x}}_{n,1}|}(\mathbf{0})$$

of width $5NL_k$ and of inner radius $|\hat{\mathbf{x}}_{n,1}|$. This explains why, for our purposes, an eigenfunction $\boldsymbol{\Phi}_n$ can be effectively "labeled" by a single localization center.

In other words, although in this paper we cannot rule out the possibility of existence of multiple centers of localization at arbitrarily large distances (depending on $\boldsymbol{\Phi}_n$ through $|\hat{\mathbf{x}}_{n,1}|$), such centers do not contribute to a "radial" quantum transport – away from the origin $\mathbf{0}$ – which might have lead to dynamical delocalization.

Lemma 3.3. *Given $k > k_0$, there exists $j_0 \geq k$ large enough such that if $j \geq j_0$, $\omega \in \Omega_k^{\mathrm{good}}$ and $\hat{\mathbf{x}}_{n,1} \in \mathbf{B}_{L_j}(\mathbf{0})$ then*

$$\left\| \left(1 - \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}(\mathbf{0})}\right) \boldsymbol{\Phi}_n \right\| \leq \frac{1}{4}.$$

Proof. By Lemma 2.4 (see also (2.10)),

$$\forall\, i \ge j+1,\ \forall\, \mathbf{w} \in \mathbb{Z}^{Nd} \setminus \boldsymbol{B}_{5NL_i}(\mathbf{0}),\ \text{the cubes } \boldsymbol{\Lambda}_{L_i}(\mathbf{w}) \text{ and } \boldsymbol{\Lambda}_{L_i}(\mathbf{0}) \text{ are separable.}$$

In addition, we take $j \ge k$, as suggested in the lemma.

Next, we divide the complement $\mathbb{R}^{Nd} \setminus \boldsymbol{\Lambda}_{5NL_{j+2}}(\mathbf{0})$ into annular regions

$$\mathbf{M}_i(\mathbf{0}) := \boldsymbol{\Lambda}_{5NL_{i+1}}(\mathbf{0}) \setminus \boldsymbol{\Lambda}_{5NL_i}(\mathbf{0}), \quad i \ge j+2, \tag{3.8}$$

and write

$$\left\| \Big(1 - \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}(\mathbf{0})}\Big) \boldsymbol{\Phi}_n \right\|^2 = \sum_{i \ge j+2} \| \mathbf{1}_{\mathbf{M}_i(\mathbf{0})} \boldsymbol{\Phi}_n \|^2 \le \sum_{i \ge j+2} \sum_{\mathbf{w} \in \mathbf{M}_i(\mathbf{0})} \| \mathbf{1}_{\mathbf{C}(\mathbf{w})} \boldsymbol{\Phi}_n \|^2.$$

Furthermore, $\hat{\mathbf{x}}_{n,1} \in \mathbf{B}_{L_j}(\mathbf{0}) \subset \mathbf{B}_{L_{i-1}}(\mathbf{0})$, so that by Lemma 3.2, the cube $\boldsymbol{\Lambda}_{L_i}(\mathbf{0})$ must be (m, E_n)-S. Therefore, by the definition of the event Ω_k^{good}, the cube $\boldsymbol{\Lambda}_{L_i}(\mathbf{w})$ is (m, E_n)-NS. Applying Lemma 2.2 to the cube $\boldsymbol{\Lambda}_{L_i}(\mathbf{w})$ and to the cell $\mathbf{C}(\mathbf{w})$, we obtain

$$\| \mathbf{1}_{\mathbf{C}(\mathbf{w})} \boldsymbol{\Phi}_n \|^2 \le \mathrm{e}^{-2mL_i}.$$

Since the volume $\big|\mathbf{M}_i(\mathbf{0})\big|$ of the annular region $\mathbf{M}_i(\mathbf{0})$ grows polynomially in L_i, the assertion of Lemma 3.3 follows. ☐

3.4. Bounds on concentration of localization centers.

Lemma 3.4. *There exists a constant $c_2 \in (0, \infty)$ such that for $\omega \in \Omega_k^{\text{bad}}$, $j \ge k$,*

$$\operatorname{card}\big\{n : \hat{\mathbf{x}}_{n,1} \in \mathbf{B}_{L_{j+1}}(\mathbf{0})\big\} \le c_2\, L_{j+1}^{\alpha N d}. \tag{3.9}$$

Proof. The left-hand-side of (3.9) is nondecreasing in j, so we can restrict ourselves to the case $j \ge j_0$. First, observe that, with $\boldsymbol{\Lambda}_{L_{j+2}} = \boldsymbol{\Lambda}_{L_{j+2}}(\mathbf{0})$

$$\sum_{n:\, \hat{\mathbf{x}}_{n,1} \in \mathbf{B}_{L_{j+1}}(\mathbf{0})} \Big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} P_I\, \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) \le \operatorname{tr}\big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} P_I \big). \tag{3.10}$$

Each term in the above sum is not less than $1/2$. Indeed,

$$\begin{aligned}
&\Big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} P_I\, \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) \\
&\quad = \Big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} P_I \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) - \Big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} P_I \big(1 - \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}}\big) \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) \\
&\quad \ge \Big(\mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}} \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) - \frac{1}{4} \qquad\qquad \text{(using Lemma 3.3)} \qquad (3.11) \\
&\quad = (\boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n) - \Big(\big(1 - \mathbf{1}_{\boldsymbol{\Lambda}_{L_{j+2}}}\big) \boldsymbol{\Phi}_n, \boldsymbol{\Phi}_n \Big) - \frac{1}{4} \\
&\quad \ge \frac{1}{2}\,. \qquad (3.12)
\end{aligned}$$

Substituting the lower bounds from (3.11) – (3.12) under the trace in Eqn (3.10), we get the desired upper bound on the LHS of Eqn (3.9). ☐

3.5. Bounds for "good" samples of potential.

Lemma 3.5. *There exists an integer $k_1 = k_1(L_0)$ such that $\forall\ k \ge k_1$, $\omega \in \Omega_{k+1}^{\text{good}}$ and* $\mathbf{x}$ *from the annular region $\mathbf{M}_{k+1}$ defined in* (3.8),

$$\Big\| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})}\ P_I\, \xi(\mathbf{H})\ \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \Big\| \le \mathrm{e}^{-mL_k/2} \|\xi\|_\infty. \tag{3.13}$$

Proof. It suffices to prove (3.13) in the particular case where $\|\xi\|_\infty \leq 1$, which we assume below. First, we bound the LHS of (3.13) as follows:

$$\begin{aligned}\| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \, P_I \, \xi(\mathbf{H}) \, \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \| &\leq \sum_{E_n \in I} |\xi(E_n)| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \\ &\leq \sum_{E_n \in I} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \end{aligned} \tag{3.14}$$

since $\|\eta\|_\infty \leq 1$. Now divide the sum according to where $\hat{\mathbf{x}}_{n,1}$ are located and write

$$\begin{aligned}\sum_{E_n \in I} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| &= \sum_{\substack{E_n \in I \\ \hat{\mathbf{x}}_{n,1} \in \boldsymbol{\Lambda}_{5NL_{k+1}}(\mathbf{0})}} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \\ &\quad + \sum_{j=k+1}^{\infty} \sum_{\substack{E_n \in I \\ \hat{\mathbf{x}}_{n,1} \in \mathbf{M}_j(\mathbf{0})}} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \|,\end{aligned}$$

with $\mathbf{M}_i(\mathbf{0})$ defined in (3.8). Since $\mathbf{x} \in \mathbf{M}_{k+1}(\mathbf{0})$, we have $\mathbf{B}_{L_k}(\mathbf{x}) \cap \mathbf{B}_{L_k}(\mathbf{0}) = \varnothing$. Then, by Lemma 2.4, the two cubes $\mathbf{B}_{L_k}(\mathbf{x})$ and $\mathbf{B}_{L_k}(\mathbf{0})$ are separable. In turn, this implies that one of these cubes is (m, E_n)-NS. Therefore, by Lemma 3.4,

$$\sum_{\substack{E_n \in I \\ \hat{\mathbf{x}}_{n,1} \in \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{0})}} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \leq c_2 \, C' \, L_{k+1}^{\alpha N d} \, \mathrm{e}^{-mL_k}.$$

Furthermore, for $k > k_0$ large enough,

$$\sum_{\substack{E_n \in I \\ \hat{\mathbf{x}}_{n,1} \in \boldsymbol{\Lambda}_{L_{k+1}}(\mathbf{0})}} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \leq \frac{1}{2} \mathrm{e}^{-mL_k/2}. \tag{3.15}$$

For any $j \geq k+2$ and $\hat{\mathbf{x}}_{n,1} \in \mathbf{M}_j(\mathbf{0})$, by Lemma 3.2, the cube $\mathbf{B}_{L_j}(\hat{\mathbf{x}}_{n,1})$ must be (m, E_n)-S, so that $\mathbf{B}_{L_j}(\mathbf{0})$ has to be (m, E_n)-NS:

$$\| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| \leq \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_j}(\mathbf{0})} \boldsymbol{\Phi}_n \| \leq C' \mathrm{e}^{-mL_j}.$$

Applying again Lemma 3.4, we see that, if $k \geq k_1$, then

$$\begin{aligned}\sum_{j=k+1}^{\infty} \sum_{\substack{E_n \in I \\ \hat{\mathbf{x}}_{n,1} \in \mathbf{M}_j(\mathbf{0})}} \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \boldsymbol{\Phi}_n \| \, \| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \boldsymbol{\Phi}_n \| &\leq C \sum_{j=k+2}^{\infty} \mathrm{e}^{-mL_j} L_j^{\alpha N d} \\ &\leq \frac{1}{2} \mathrm{e}^{-mL_k/2}.\end{aligned}$$

Combining this estimate with (3.14) and (3.15), the assertion of Lemma 3.5 follows. □

3.6. **Bounds for "bad" samples of potential.**

Lemma 3.6. *Let k_1 be as in Lemma 3.5 and assume that $k \geq k_1$ and $\mathbf{x} \in \mathbf{M}_{k+1}(\mathbf{0})$. We have:*

$$\mathbb{E}\left[\left\| \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{x})} \, P_I \, \xi(\mathbf{H}) \, \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})} \right\| \right] \leq \|\xi\|_\infty \left(C L_k^{-2p+2Nd\alpha} + \mathrm{e}^{-mL_k/2} \right).$$

Proof. We again assume $\|\xi\|_\infty \le 1$. For $\omega \in \Omega_k^{\rm bad}$ we can use Sect. 3.1 while for $\omega \in \Omega_k^{\rm good}$ we can use Sect. 3.5. More precisely, the above expectation is bounded by

$$\mathbb{P}\left\{\Omega_k^{\rm bad}\right\} + \mathrm{e}^{-mL_k/2}\,\mathbb{P}\left\{\Omega_k^{\rm good}\right\} \le CL_k^{-2p+2Nd\alpha} + \mathrm{e}^{-mL_k/2}. \qquad \square$$

3.7. **Conclusion.** For a compact subset $\mathbf{K} \subset \mathbb{R}^{Nd}$ we find an integer $k \ge k_1$ such that $\mathbf{K} \subset \boldsymbol{\Lambda}_{L_k}(\mathbf{0})$. Then, with the annular regions $\mathbf{M}_j(\mathbf{0})$,

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{X}^Q\, P_I\, \xi(\mathbf{H}(\omega))\; \mathbf{1}_{\mathbf{K}}\,\|\right] &\le L_k^Q + \sum_{j\ge k+1} \mathbb{E}\left[\|\mathbf{X}^Q\; \mathbf{1}_{\mathbf{M}_j(\mathbf{0})}\, P_I\, \xi(\mathbf{H})\; \mathbf{1}_{\mathbf{K}}\,\|\right]\\ &\le L_k^Q + \sum_{j\ge k+1} L_{j+1}^Q \left(\sum_{\mathbf{w}\in\mathbf{M}_j(\mathbf{0})} \mathbb{E}\left[\|\, \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{w})}\, P_I\, \xi(\mathbf{H})\; \mathbf{1}_{\boldsymbol{\Lambda}_{L_k}(\mathbf{0})}\,\|\right]\right)\\ &\le L_k^Q + \sum_{j\ge k+1} L_j^{\alpha Q} L_j^{Nd\alpha}\left(L_j^{-2p+2Nd\alpha} + \mathrm{e}^{-mL_j/2}\right) < \infty,\end{aligned}$$

since $2p > 3Nd\alpha + \alpha Q$ by assumption (3.2), and $L_j \sim \left[L_0^{\alpha^j}\right]$ grow fast enough.

This completes the proof of dynamical localization. $\square$

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