5 Linear filters

5.1 The Filter Theorem

A linear filter of one random sequence $\{X_t\}$ into another sequence $\{Y_t\}$ is

$$Y_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s} \,. \tag{5.1}$$

THEOREM 5.1 (the filter theorem) Suppose X_t is a stationary time series with spectral density $f_X(\omega)$. Let $\{a_t\}$ be a sequence of real numbers such that $\sum_{t=-\infty}^{\infty} |a_t| < \infty$. Then the process $Y_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s}$ is a stationary time series with spectral density function

$$f_Y(\omega) = \left|A(e^{i\omega})\right|^2 f_X(\omega) = |a(\omega)|^2 f_X(\omega)\,,$$

where $A(z)$ is the **filter generating function**

$$A(z) = \sum_{s=-\infty}^{\infty} a_s z^s, \quad |z| \le 1\,.$$

and $a(\omega) = A(e^{i\omega})$ is the **transfer function** of the linear filter.

Proof.

$$\begin{aligned}
\operatorname{cov}(Y_t, Y_{t+k}) &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_r a_s \operatorname{cov}(X_{t-r}, X_{t+k-s}) \\
&= \sum_{r,s\in\mathbb{Z}} a_r a_s \gamma_{k+r-s} \\
&= \sum_{r,s\in\mathbb{Z}} a_r a_s \int_{-\pi}^{\pi} \tfrac{1}{2} e^{i\omega(k+r-s)} f_X(\omega) d\omega \\
&= \int_{-\pi}^{\pi} A(e^{i\omega}) A(e^{-i\omega}) \tfrac{1}{2} e^{i\omega k} f_X(\omega) d\omega \\
&= \int_{-\pi}^{\pi} \tfrac{1}{2} e^{i\omega k} \left|A(e^{i\omega})\right|^2 f_X(\omega) d\omega \\
&= \int_{-\pi}^{\pi} \tfrac{1}{2} e^{i\omega k} f_Y(\omega) d\omega\,.
\end{aligned}$$

Thus $f_Y(\omega)$ is the spectral density for Y and Y is stationary. ■

5.2 Application to autoregressive processes

Let us use the notation B for the **backshift operator**

$$B^0 = I, \quad (B^0X)_t = X_t, \quad (BX)_t = X_{t-1}, \quad (B^2X)_t = X_{t-2}, \quad \dots$$

Then the AR(p) process can be written as

$$\left(I - \sum_{r=1}^{p} \phi_r B^r\right) X = \epsilon$$

or $\phi(B)X = \epsilon$, where ϕ is the function

$$\phi(z) = 1 - \sum_{r=1}^{p} \phi_r z^r\,.$$

By the filter theorem, $f_\epsilon(\omega) = |\phi\left(e^{i\omega}\right)|^2 f_X(\omega)$, so since $f_\epsilon(\omega) = \sigma^2/\pi$,

$$f_X(\omega) = \frac{\sigma^2}{\pi|\phi(e^{i\omega})|^2}\,. \tag{5.2}$$

As $f_X(\omega) = (1/\pi)\sum_{k=-\infty}^{\infty} \gamma_k e^{i\omega k}$, we can calculate the autocovariances by expanding $f_X(\omega)$ as a power series in $e^{i\omega}$. For this to work, the zeros of $\phi(z)$ must lie outside the unit circle in $\mathbb{C}$. This is the stationarity condition for the AR(p) process.

EXAMPLE 5.2
For the AR(1) process, $X_t - \phi_1 X_{t-1} = \epsilon_t$, we have $\phi(z) = 1 - \phi_1 z$, with its zero at $z = 1/\phi_1$. The stationarity condition is $|\phi_1| < 1$. Using (5.2) we find

$$f_X(\omega) = \frac{\sigma^2}{\pi|1-\phi e^{i\omega}|^2} = \frac{\sigma^2}{\pi(1 - 2\phi\cos\omega + \phi^2)}\,,$$

which is what we found by other another method in Example 3.1(c). To find the autocovariances we can write

$$\begin{aligned}
\frac{1}{|\phi(z)|^2} = \frac{1}{\phi(z)\phi(1/z)} &= \frac{1}{(1-\phi_1 z)(1-\phi_1/z)} = \sum_{r=0}^{\infty}\phi_1^r z^r \sum_{s=0}^{\infty}\phi_1^s z^{-s} \\
&= \sum_{k=-\infty}^{\infty} z^k(\phi_1^{|k|}(1 + \phi_1^2 + \phi_1^4 + \cdots)) = \sum_{k=-\infty}^{\infty} \frac{z^k\phi_1^{|k|}}{1-\phi_1^2} \\
&\implies f_X(\omega) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{\sigma^2\phi_1^{|k|}}{1-\phi_1^2} e^{i\omega k}
\end{aligned}$$

and so $\gamma_k = \sigma^2\phi_1^{|k|}/(1-\phi_1^2)$ as we saw before.

In general, it is often easier to calculate the spectral density function first, using filters, and then deduce the autocovariance function from it.

5.3 Application to moving average processes

The MA(q) process $X_t = \epsilon_t + \sum_{s=1}^{q} \theta_s \epsilon_{t-s}$ can be written as

$$X = \theta(B)\epsilon$$

where $\theta(z) = \sum_{s=0}^{q} \theta_s B^s$. By the filter theorem, $f_X(\omega) = |\theta(e^{i\omega})|^2(\sigma^2/\pi)$.

Example 5.3
For the MA(1), $X_t = \epsilon_t + \theta_1 \epsilon_{t-1}$, $\theta(z) = 1 + \theta_1 z$ and

$$f_X(\omega) = \frac{\sigma^2}{\pi}\left(1 + 2\theta_1 \cos\omega + \theta_1^2\right) .$$

As above, we can obtain the autocovariance function by expressing $f_X(\omega)$ as a power series in $e^{i\omega}$. We have

$$f_X(\omega) = \frac{\sigma^2}{\pi}\left(\theta_1 e^{-i\omega} + (1+\theta_1^2) + \theta_1 e^{i\omega}\right) = \frac{\sigma^2}{\pi}\theta(e^{i\omega})\theta(e^{-i\omega})$$

So $\gamma_0 = \sigma^2(1+\theta_1^2)$, $\gamma_1 = \theta_1\sigma^2$, $\gamma_2 = 0$, $|k| > 1$.

As we remarked in Section 1.5, the autocovariance function of a MA(1) process with parameters (σ^2, θ_1) is identical to one with parameters $(\theta_1^2\sigma^2, \theta_1^{-1})$. That is,

$$\begin{aligned} \gamma_0^* &= \theta_1^2\sigma^2(1 + 1/\theta_1^2) = \sigma^2(1+\theta_1^2) = \gamma_0 \\ \rho_1^* &= \theta_1^{-1}/(1+\theta_1^{-2}) = \theta_1/(1+\theta_1^2) = \rho_1 . \end{aligned}$$

In general, the MA(q) process can be written as $X = \theta(B)\epsilon$, where

$$\theta(z) = \sum_{k=0}^{q} \theta_k z^k = \prod_{k=1}^{q}(\omega_k - z) .$$

So the autocovariance generating function is

$$g(z) = \sum_{k=-q}^{q} \gamma_k z^k = \sigma^2\theta(z)\theta(z^{-1}) = \sigma^2 \prod_{k=1}^{q}(\omega_k - z)(\omega_k - z^{-1}) . \tag{5.3}$$

Note that $(\omega_k - z)(\omega_k - z^{-1}) = \omega_k^2(\omega_k^{-1} - z)(\omega_k^{-1} - z^{-1})$. So $g(z)$ is unchanged in (5.3) if (for any i such that ω_k is real) we replace ω_k by ω_k^{-1} and multiply σ^2 by ω_k^2. Thus (if all roots of $\theta(z) = 0$ are real) there can be 2^q different $MA(q)$ processes with the same autocovariance function. For **identifiability**, we assume that all the roots of $\theta(z)$ lie outside the unit circle in $\mathbb{C}$. This is equivalent to the invertibility condition, that ϵ_t can be written as a convergent power series in $\{X_t, X_{t-1}, \dots\}$.

5.4 The general linear process

A special case of (5.1) is the **general linear process**,

$$Y_t = \sum_{s=0}^{\infty} a_s X_{t-s} ,$$

where $\{X_t\}$ is white noise. This has

$$\mathrm{cov}(Y_t, Y_{t+k}) = \sigma^2 \sum_{s=0}^{\infty} a_s a_{s+k} \le \sigma^2 \sum_{s=0}^{\infty} a_s^2 ,$$

where the inequality is an equality when $k = 0$. Thus $\{Y_t\}$ is stationary if and only if $\sum_{s=0}^{\infty} a_s^2 < \infty$. In practice the general linear model is useful when the a_s are expressible in terms of a finite number of parameters which can be estimated. A rich class of such models are the ARMA models.

5.5 Filters and ARMA processes

The ARMA(p,q) model can be written as $\phi(B)X = \theta(B)\epsilon$. Thus

$$|\phi(e^{i\omega})|^2 f_X(\omega) = |\theta(e^{i\omega})|^2 \frac{\sigma^2}{\pi} \quad \Longrightarrow \quad f_X(\omega) = \left|\frac{\theta(e^{i\omega})}{\phi(e^{i\omega})}\right|^2 \frac{\sigma^2}{\pi} .$$

This is subject to the conditions that

- the zeros of ϕ lie outside the unit circle in $\mathbb{C}$ for stationarity.
- the zeros of θ lie outside the unit circle in $\mathbb{C}$ for identifiability.
- $\phi(z)$ and $\theta(z)$ have no common roots.

If there were a common root, say $1/\alpha$, so that $(I - \alpha B)\phi_1(B)X = (I - \alpha B)\theta_1(B)\epsilon$, then we could multiply both sides by $\sum_{n=0}^{\infty}\alpha^n B^n$ and deduce $\phi_1(B)X = \theta_1(B)\epsilon$, and thus that a more economical ARMA($p-1, q-1$) model suffices.

5.6 Calculating autocovariances in ARMA models

As above, the filter theorem can assist in calculating the autocovariances of a model. These can be compared with autocovariances estimated from the data. For example, an ARMA(1, 2) has

$$\phi(z) = 1 - \phi z, \quad \theta(z) = 1 + \theta_1 z + \theta_2 z^2, \quad \text{where } |\phi| < 1.$$

Then $X = C(B)\epsilon$, where

$$C(z) = \theta(z)/\phi(z) = \left(1 + \theta_1 z + \theta_2 z^2\right) \sum_{n=0}^{\infty} \phi^n z^n = \sum_{n=0}^{\infty} c_n z^n ,$$

with $c_0 = 1$, $c_1 = \phi + \theta_1$, and

$$c_n = \phi^n + \phi^{n-1}\theta_1 + \phi^{n-1}\theta_2 = \phi^{n-2}\left(\phi^2 + \phi\theta_1 + \theta^2\right), \quad n \ge 2.$$

So $X_t = \sum_{n=0}^{\infty} c_n \epsilon_{t-n}$ and we can compute covariances as

$$\gamma_k = \mathrm{cov}(X_t, X_{t+k}) = \sum_{n,m=0}^{\infty} c_n c_m \,\mathrm{cov}(\epsilon_{t-n}, \epsilon_{t+k-m}) = \sum_{n=0}^{\infty} c_n c_{n+k} \sigma^2 .$$

For example, $\gamma_k = \phi\gamma_{k-1}$, $k \ge 3$. As a test of whether the model is ARMA(1, 2) we might look to see if the sample autocovariances decay geometrically, for $k \ge 2$,