4 Estimation of the spectrum

4.1 The periodogram

Suppose we have $T = 2m + 1$ observations of a time series, $y_1, \dots, y_T$. Define the **Fourier frequencies**, $\omega_j = 2\pi j/T$, $j = 1, \dots, m$, and consider the regression model

$$y_t = \alpha_0 + \sum_{j=1}^{m} \alpha_j \cos(\omega_j t) + \sum_{j=1}^{m} \beta_j \sin(\omega_j t)\,,$$

which can be written as a general linear model, $Y = X\theta + \epsilon$, where

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix}, \quad X = \begin{pmatrix} 1 & c_{11} & s_{11} & \cdots & c_{m1} & s_{m1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & c_{1T} & s_{1T} & \cdots & c_{mT} & s_{mT} \end{pmatrix}, \quad \theta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_m \\ \beta_m \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_T \end{pmatrix},$$

$$c_{jt} = \cos(\omega_j t), \qquad s_{jt} = \sin(\omega_j t)\,.$$

The least squares estimates in this model are given by

$$\hat\theta = (X^\top X)^{-1} X^\top Y\,.$$

Note that

$$\sum_{t=1}^{T} e^{i\omega_j t} = \frac{e^{i\omega_j}(1 - e^{i\omega_j T})}{1 - e^{i\omega_j}} = 0$$

$$\implies \sum_{t=1}^{T} c_{jt} + i \sum_{t=1}^{T} s_{jt} = 0 \quad \implies \quad \sum_{t=1}^{T} c_{jt} = \sum_{t=1}^{T} s_{jt} = 0$$

and

$$\sum_{t=1}^{T} c_{jt} s_{jt} = \tfrac{1}{2} \sum_{t=1}^{T} \sin(2\omega_j t) = 0\,,$$

$$\sum_{t=1}^{T} c_{jt}^2 = \tfrac{1}{2} \sum_{t=1}^{T} \{1 + \cos(2\omega_j t)\} = T/2\,,$$

$$\sum_{t=1}^{T} s_{jt}^2 = \tfrac{1}{2} \sum_{t=1}^{T} \{1 - \cos(2\omega_j t)\} = T/2\,,$$

$$\sum_{t=1}^{T} c_{jt} s_{kt} = \sum_{t=1}^{T} c_{jt} c_{kt} = \sum_{t=1}^{T} s_{jt} s_{kt} = 0, \quad j \neq k\,.$$

Using these, we have

$$\hat\theta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} T & 0 & \cdots & 0 \\ 0 & T/2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T/2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_t y_t \\ \sum_t c_{1t} y_t \\ \vdots \\ \sum_t s_{mt} y_t \end{pmatrix} = \begin{pmatrix} \bar y \\ (2/T) \sum_t c_{1t} y_t \\ \vdots \\ (2/T) \sum_t s_{mt} y_t \end{pmatrix}$$

and the regression sum of squares is

$$\hat Y^\top \hat Y = Y^\top X (X^\top X)^{-1} X^\top Y = T\bar y^2 + \sum_{j=1}^{m} \frac{2}{T} \left[\left\{ \sum_{t=1}^{T} c_{jt} y_t \right\}^2 + \left\{ \sum_{t=1}^{T} s_{jt} y_t \right\}^2 \right].$$

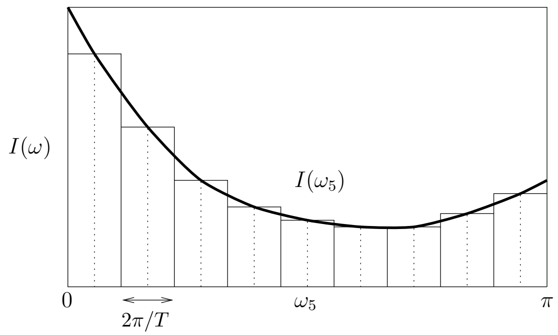
Since we are fitting T unknown parameters to T data points, the model fits with no residual error, i.e., $\hat Y = Y$. Hence

$$\sum_{t=1}^{T} (y_t - \bar y)^2 = \sum_{j=1}^{m} \frac{2}{T} \left[\left\{ \sum_{t=1}^{T} c_{jt} y_t \right\}^2 + \left\{ \sum_{t=1}^{T} s_{jt} y_t \right\}^2 \right].$$

This motivates definition of the **periodogram** as

$$I(\omega) = \frac{1}{\pi T} \left[\left\{ \sum_{t=1}^{T} y_t \cos(\omega t) \right\}^2 + \left\{ \sum_{t=1}^{T} y_t \sin(\omega t) \right\}^2 \right].$$

A factor of $(1/2\pi)$ has been introduced into this definition so that the sample variance, $\hat\gamma_0 = (1/T) \sum_{t=1}^{T} (y_t - \bar y)^2$, equates to the sum of the areas of m rectangles, whose heights are $I(\omega_1), \dots, I(\omega_m)$, whose widths are $2\pi/T$, and whose bases are centred at $\omega_1, \dots, \omega_m$. I.e., $\hat\gamma_0 = (2\pi/T) \sum_{j=1}^{m} I(\omega_j)$. These rectangles approximate the area under the curve $I(\omega)$, $0 \le \omega \le \pi$.



Using the fact that $\sum_{t=1}^{T} c_{jt} = \sum_{t=1}^{T} s_{jt} = 0$, we can write

$$\begin{aligned}
\pi T I(\omega_j) &= \left\{\sum_{t=1}^{T} y_t \cos(\omega_j t)\right\}^2 + \left\{\sum_{t=1}^{T} y_t \sin(\omega_j t)\right\}^2 \\
&= \left\{\sum_{t=1}^{T} (y_t - \bar{y}) \cos(\omega_j t)\right\}^2 + \left\{\sum_{t=1}^{T} (y_t - \bar{y}) \sin(\omega_j t)\right\}^2 \\
&= \left|\sum_{t=1}^{T} (y_t - \bar{y}) e^{i\omega_j t}\right|^2 \\
&= \sum_{t=1}^{T} (y_t - \bar{y}) e^{i\omega_j t} \sum_{s=1}^{T} (y_s - \bar{y}) e^{-i\omega_j s} \\
&= \sum_{t=1}^{T} (y_t - \bar{y})^2 + 2 \sum_{k=1}^{T-1} \sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y}) \cos(\omega_j k) .
\end{aligned}$$

Hence

$$I(\omega_j) = \frac{1}{\pi} \hat{\gamma}_0 + \frac{2}{\pi} \sum_{k=1}^{T-1} \hat{\gamma}_k \cos(\omega_j k) .$$

$I(\omega)$ is therefore a sample version of the spectral density $f(\omega)$.

4.2 Distribution of spectral estimates

If the process is stationary and the spectral density exists then $I(\omega)$ is an almost unbiased estimator of $f(\omega)$, but it is a rather poor estimator without some smoothing.

Suppose $\{y_t\}$ is Gaussian white noise, i.e., $y_1, \ldots, y_T$ are iid $N(0, \sigma^2)$. Then for any Fourier frequency $\omega = 2\pi j/T$,

$$I(\omega) = \frac{1}{\pi T} \left[A(\omega)^2 + B(\omega)^2\right] , \tag{4.1}$$

where

$$A(\omega) = \sum_{t=1}^{T} y_t \cos(\omega t) , \qquad B(\omega) = \sum_{t=1}^{T} y_t \sin(\omega t) . \tag{4.2}$$

Clearly $A(\omega)$ and $B(\omega)$ have zero means, and

$$\begin{aligned}
\operatorname{var}[A(\omega)] &= \sigma^2 \sum_{t=1}^{T} \cos^2(\omega t) = T\sigma^2/2 , \\
\operatorname{var}[B(\omega)] &= \sigma^2 \sum_{t=1}^{T} \sin^2(\omega t) = T\sigma^2/2 ,
\end{aligned}$$

$$\operatorname{cov}[A(\omega), B(\omega)] = \mathbb{E}\left[\sum_{t=1}^{T} \sum_{s=1}^{T} y_t y_s \cos(\omega t) \sin(\omega s)\right] = \sigma^2 \sum_{t=1}^{T} \cos(\omega t) \sin(\omega t) = 0 .$$

Hence $A(\omega)\sqrt{2/T\sigma^2}$ and $B(\omega)\sqrt{2/T\sigma^2}$ are independently distributed as $N(0,1)$, and $2\left[A(\omega)^2 + B(\omega)^2\right]/(T\sigma^2)$ is distributed as χ_2^2. This gives $I(\omega) \sim (\sigma^2/\pi)\chi_2^2/2$. Thus we see that $I(w)$ is an unbiased estimator of the spectrum, $f(\omega) = \sigma^2/\pi$, but it is not consistent, since $\operatorname{var}[I(\omega)] = \sigma^4/\pi^2$ does not tend to 0 as $T \to \infty$. This is perhaps surprising, but is explained by the fact that as T increases we are attempting to estimate $I(\omega)$ for an increasing number of Fourier frequencies, with the consequence that the precision of each estimate does not change.

By a similar argument, we can show that for any two Fourier frequencies, ω_j and ω_k the estimates $I(\omega_j)$ and $I(\omega_k)$ are statistically independent. These conclusions hold more generally.

Theorem 4.1 Let $\{Y_t\}$ be a stationary Gaussian process with spectrum $f(\omega)$. Let $I(\cdot)$ be the periodogram based on samples $Y_1, \ldots, Y_T$, and let $\omega_j = 2\pi j/T$, $j < T/2$, be a Fourier frequency. Then in the limit as $T \to \infty$,

(a) $I(\omega_j) \sim f(\omega_j)\chi_2^2/2$.

(b) $I(\omega_j)$ and $I(\omega_k)$ are independent for $j \neq k$.

Assuming that the underlying spectrum is smooth, $f(\omega)$ is nearly constant over a small range of ω. This motivates use of an estimator for the spectrum of

$$\hat{f}(\omega_j) = \frac{1}{2p+1} \sum_{\ell=-p}^{p} I(\omega_{j+\ell}) .$$

Then $\hat{f}(\omega_j) \sim f(\omega_j)\chi^2_{2(2p+1)}/[2(2p+1)]$, which has variance $f(\omega)^2/(2p+1)$. The idea is to let $p \to \infty$ as $T \to \infty$.

4.3 The fast Fourier transform

$I(\omega_j)$ can be calculated from (4.1)–(4.2), or from

$$I(\omega_j) = \frac{1}{\pi T} \left|\sum_{t=1}^{T} y_t e^{i\omega_j t}\right|^2 .$$

Either way, this requires of order T multiplications. Hence to calculate the complete periodogram, i.e., $I(\omega_1), \ldots, I(\omega_m)$, requires of order T^2 multiplications. Computation effort can be reduced significantly by use of the **fast Fourier transform**, which computes $I(\omega_1), \ldots, I(\omega_m)$ using only order $T \log_2 T$ multiplications.