3 Spectral methods

3.1 The discrete Fourier transform

If $h(t)$ is defined for integers t, the discrete Fourier transform of h is

$$H(\omega) = \sum_{t=-\infty}^{\infty} h(t)e^{-i\omega t}, \quad -\pi \le \omega \le \pi$$

The inverse transform is

$$h(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega t} H(\omega)\, d\omega\, .$$

If $h(t)$ is real-valued, and an even function such that $h(t) = h(-t)$, then

$$H(\omega) = h(0) + 2\sum_{t=1}^{\infty} h(t)\cos(\omega t)$$

and

$$h(t) = \frac{1}{\pi}\int_{0}^{\pi} \cos(\omega t) H(\omega)\, d\omega\, .$$

3.2 The spectral density

The Wiener-Khintchine theorem states that for any real-valued stationary process there exists a **spectral distribution function**, $F(\cdot)$, which is a nondecreasing and right continuous on $[0, \pi]$ such that $F(0) = 0$, $F(\pi) = \gamma_0$ and

$$\gamma_k = \int_0^{\pi} \cos(\omega k)\, dF(\omega)\, .$$

The integral is a Lebesgue-Stieltges integral and is defined even if F has discontinuities. Informally, $F(\omega_2) - F(\omega_1)$ is the contribution to the variance of the series made by frequencies in the range (ω_1, ω_2).

$F(\cdot)$ can have jump discontinuities, but always can be decomposed as

$$F(\omega) = F_1(\omega) + F_2(\omega)$$

where $F_1(\cdot)$ is a nondecreasing continuous function and $F_2(\cdot)$ is a nondecreasing step function. This is a decomposition of the series into a purely indeterministic component and a deterministic component.

Suppose the process is purely indeterministic, (which happens if and only if $\sum_k |\gamma_k| < \infty$). In this case $F(\cdot)$ is a nondecreasing continuous function, and differentiable at all points (except possibly on a set of measure zero). Its its derivative

$f(\omega) = F'(\omega)$ exists, and is called the **spectral density function**. Apart from a multiplication by $1/\pi$ it is simply the discrete Fourier transform of the autocovariance function and is given by

$$f(\omega) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\omega} = \frac{1}{\pi}\gamma_0 + \frac{2}{\pi}\sum_{1}^{\infty}\gamma_k \cos(\omega k)\, ,$$

with inverse

$$\gamma_k = \int_0^{\pi} \cos(\omega k) f(\omega)\, d\omega\, .$$

Note. Some authors define the spectral distribution function on $[-\pi, \pi]$; the use of negative frequencies makes the interpretation of the spectral distribution less intuitive and leads to a difference of a factor of 2 in the definition of the spectra density. Notice, however, that if f is defined as above and extended to negative frequencies, $f(-\omega) = f(\omega)$, then we can write

$$\gamma_k = \int_{-\pi}^{\pi} \tfrac{1}{2} e^{i\omega k} f(\omega)\, d\omega\, .$$

Example 3.1

(a) Suppose $\{X_t\}$ is i.i.d., $\gamma_0 = \mathrm{var}(X_t) = \sigma^2 > 0$ and $\gamma_k = 0$, $k \ge 1$. Then $f(\omega) = \sigma^2/\pi$. The fact that the spectral density is flat means that all frequencies are equally present accounts for our calling this sequence **white noise**.

(b) As an example of a process which in not purely indeterministic, consider $X_t = \cos(\omega_0 t + U)$ where ω_0 is a value in $[0, \pi]$ and $U \sim U[-\pi, \pi]$. The process has zero mean, since

$$\mathbb{E}(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + u)\, du = 0$$

and autocovariance

$$\begin{aligned}
\gamma_k &= \mathbb{E}(X_t, X_{t+k}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega_0 t + u)\cos(\omega_0 t + \omega_0 k + u) du \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\left[\cos(\omega_0 k) + \cos(2\omega_0 t + \omega_0 k + 2u)\right] du \\
&= \frac{1}{2\pi}\tfrac{1}{2}[2\pi\cos(\omega_0 k) + 0] \\
&= \frac{1}{2}\cos(\omega_0 k)\, .
\end{aligned}$$

Hence X_t is second order stationary and we have

$$\gamma_k = \frac{1}{2}\cos(\omega_0 k), \quad F(\omega) = \frac{1}{2} I_{[\omega \geq \omega_0]} \quad \text{and} \quad f(\omega) = \frac{1}{2}\delta_{\omega_0}(\omega)\,.$$

Note that F is a nondecreasing step function.

More generally, the spectral density

$$f(\omega) = \sum_{j=1}^{n} \frac{1}{2} a_j \delta_{\omega_j}(\omega)$$

corresponds to the process $X_t = \sum_{j=1}^{n} a_j \cos(\omega_j t + U_j)$ where $\omega_j \in [0, \pi]$ and $U_1, \ldots, U_n$ are i.i.d. $U[-\pi, \pi]$.

(c) The MA(1) process, $X_t = \theta_1 \epsilon_{t-1} + \epsilon_t$, where $\{\epsilon_t\}$ is white noise. Recall $\gamma_0 = (1+\theta_1^2)\sigma^2$, $\gamma_1 = \theta_1 \sigma^2$, and $\gamma_k = 0$, $k > 1$. Thus

$$f(\omega) = \frac{\sigma^2(1 + 2\theta_1 \cos\omega + \theta_1^2)}{\pi}\,.$$

(d) The AR(1) process, $X_t = \phi_1 X_{t-1} + \epsilon_t$, where $\{\epsilon_t\}$ is white noise. Recall

$$\operatorname{var}(X_t) = \phi_1^2 \operatorname{var}(X_{t-1}) + \sigma^2 \implies \gamma_0 = \phi_1^2 \gamma_0 + \sigma^2 \implies \gamma_0 = \sigma^2/(1-\phi_1^2)$$

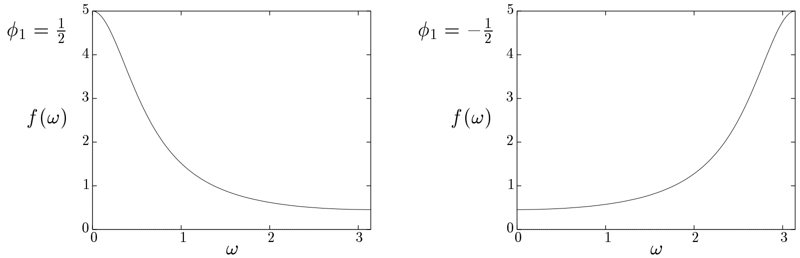
where we need $|\phi_1| < 1$ for X_t stationary. Also,

$$\gamma_k = \operatorname{cov}(X_t, X_{t-k}) = \operatorname{cov}(\phi_1 X_{t-1} + \epsilon_t, X_{t-k}) = \phi_1 \gamma_{k-1}.$$

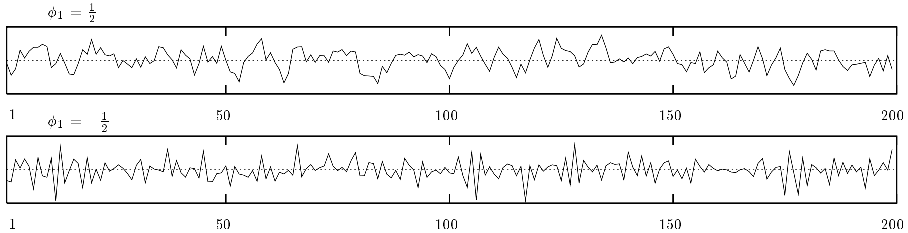
So $\gamma_k = \phi_1^{|k|}\gamma_0$, $k \in \mathbb{Z}$. Thus

$$\begin{aligned} f(\omega) &= \frac{\gamma_0}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \phi_1^k \gamma_0 \cos(k\omega) = \frac{\gamma_0}{\pi}\left\{1 + \sum_{k=1}^{\infty} \phi_1^k \left[e^{i\omega k} + e^{-i\omega k}\right]\right\} \\ &= \frac{\gamma_0}{\pi}\left\{1 + \frac{\phi_1 e^{i\omega}}{1-\phi_1 e^{i\omega}} + \frac{\phi_1 e^{-i\omega}}{1-\phi_1 e^{-i\omega}}\right\} = \frac{\gamma_0}{\pi}\frac{1-\phi_1^2}{1 - 2\phi_1\cos\omega + \phi_1^2} \\ &= \frac{\sigma^2}{\pi(1 - 2\phi_1 \cos\omega + \phi_1^2)}\,. \end{aligned}$$

Note that $\phi > 0$ has power at low frequency, whereas $\phi < 0$ has power at high frequency.



Plots above are the spectral densities for AR(1) processes in which $\{\epsilon_t\}$ is Gaussian white noise, with $\sigma^2/\pi = 1$. Samples for 200 data points are shown below.



3.3 Analysing the effects of smoothing

Let $\{a_s\}$ be a sequence of real numbers. A **linear filter** of $\{X_t\}$ is

$$Y_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s}\,.$$

In Chapter 5 we show that the spectral density of $\{Y_t\}$ is given by

$$f_Y(\omega) = |a(\omega)|^2 f_X(\omega)\,,$$

where $a(z)$ is the **transfer function**

$$a(\omega) = \sum_{s=-\infty}^{\infty} a_s e^{i\omega s}\,.$$

This result can be used to explore the effect of smoothing a series.

Example 3.2

Suppose the AR(1) series above, with $\phi_1 = -0.5$, is smoothed by a moving average on three points, so that smoothed series is

$$Y_t = \tfrac{1}{3}[X_{t+1} + X_t + X_{t-1}]\,.$$

Then $|a(\omega)|^2 = |\frac{1}{3}e^{-i\omega} + \frac{1}{3} + \frac{1}{3}e^{i\omega}|^2 = \frac{1}{9}(1 + 2\cos\omega)^2$.

Notice that $\gamma_X(0) = 4\pi/3$, $\gamma_Y(0) = 2\pi/9$, so $\{Y_t\}$ has 1/6 the variance of $\{X_t\}$. Moreover, all components of frequency $\omega = 2\pi/3$ (i.e., period 3) are eliminated in the smoothed series.