1 Models for time series

1.1 Time series data

A time series is a set of statistics, usually collected at regular intervals. Time series data occurs naturally in many application areas.

- economics - e.g., monthly data for unemployment, hospital admissions, etc.
- finance - e.g., daily exchange rate, a share price, etc.
- environmental - e.g., daily rainfall, air quality readings.
- medicine - e.g., ECG brain wave activity every 2^{-8} secs.

The methods of time series analysis pre-date those for general stochastic processes and Markov Chains. The aims of time series analysis are to describe and summarise time series data, fit low-dimensional models, and make forecasts.

We write our real-valued series of observations as $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$, a doubly infinite sequence of real-valued random variables indexed by $\mathbb{Z}$.

1.2 Trend, seasonality, cycles and residuals

One simple method of describing a series is that of **classical decomposition**. The notion is that the series can be decomposed into four elements:

Trend (T_t) — long term movements in the mean;

Seasonal effects (I_t) — cyclical fluctuations related to the calendar;

Cycles (C_t) — other cyclical fluctuations (such as a business cycles);

Residuals (E_t) — other random or systematic fluctuations.

The idea is to create separate models for these four elements and then combine them, either additively

$$X_t = T_t + I_t + C_t + E_t$$

or multiplicatively

$$X_t = T_t \cdot I_t \cdot C_t \cdot E_t\,.$$

1.3 Stationary processes

1. A sequence $\{X_t, t \in \mathbb{Z}\}$ is **strongly stationary** or **strictly stationary** if

$$(X_{t_1}, \ldots, X_{t_k}) \stackrel{\mathcal{D}}{=} (X_{t_1+h}, \ldots, X_{t_k+h})$$

for all sets of time points $t_1, \ldots, t_k$ and integer h.

2. A sequence is **weakly stationary**, or **second order stationary** if

   (a) $\mathbb{E}(X_t) = \mu$, and

   (b) $\operatorname{cov}(X_t, X_{t+k}) = \gamma_k$,

   where μ is constant and γ_k is independent of t.

3. The sequence $\{\gamma_k,\ k \in \mathbb{Z}\}$ is called the **autocovariance function**.

4. We also define

$$\rho_k = \gamma_k/\gamma_0 = \operatorname{corr}(X_t, X_{t+k})$$

   and call $\{\rho_k, k \in \mathbb{Z}\}$ the **autocorrelation function** (ACF).

Remarks.

1. A strictly stationary process is weakly stationary.
2. If the process is Gaussian, that is $(X_{t_1}, \ldots, X_{t_k})$ is multivariate normal, for all $t_1, \ldots, t_k$, then weak stationarity implies strong stationarity.
3. $\gamma_0 = \operatorname{var}(X_t) > 0$, assuming X_t is genuinely random.
4. By symmetry, $\gamma_k = \gamma_{-k}$, for all k.

1.4 Autoregressive processes

The **autoregressive process** of order p is denoted AR(p), and defined by

$$X_t = \sum_{r=1}^{p} \phi_r X_{t-r} + \epsilon_t \tag{1.1}$$

where $\phi_1, \ldots, \phi_r$ are fixed constants and $\{\epsilon_t\}$ is a sequence of independent (or uncorrelated) random variables with mean 0 and variance σ^2.

The AR(1) process is defined by

$$X_t = \phi_1 X_{t-1} + \epsilon_t\,. \tag{1.2}$$

To find its autocovariance function we make successive substitutions, to get

$$X_t = \epsilon_t + \phi_1(\epsilon_{t-1} + \phi_1(\epsilon_{t-2} + \cdots)) = \epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + \cdots$$

The fact that $\{X_t\}$ is second order stationary follows from the observation that $\mathbb{E}(X_t) = 0$ and that the autocovariance function can be calculated as follows:

$$\gamma_0 = \mathbb{E}\left(\epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + \cdots\right)^2 = \left(1 + \phi_1^2 + \phi_1^4 + \cdots\right)\sigma^2 = \frac{\sigma^2}{1-\phi_1^2}$$

$$\gamma_k = \mathbb{E}\left(\sum_{r=0}^{\infty} \phi_1^r \epsilon_{t-r} \sum_{s=0}^{\infty} \phi_1^s \epsilon_{t+k-s}\right) = \frac{\sigma^2\phi_1^k}{1-\phi_1^2}\,.$$

There is an easier way to obtain these results. Multiply equation (1.2) by X_{t-k} and take the expected value, to give

$$\mathbb{E}(X_t X_{t-k}) = \mathbb{E}(\phi_1 X_{t-1} X_{t-k}) + \mathbb{E}(\epsilon_t X_{t-k}) .$$

Thus $\gamma_k = \phi_1 \gamma_{k-1}$, $k = 1, 2, \ldots$

Similarly, squaring (1.2) and taking the expected value gives

$$\mathbb{E}(X_t^2) = \phi_1 \mathbb{E}(X_{t-1}^2) + 2\phi_1 \mathbb{E}(X_{t-1}\epsilon_t) + \mathbb{E}(\epsilon_t^2) = \phi_1^2 \mathbb{E}(X_{t-1}^2) + 0 + \sigma^2$$

and so $\gamma_0 = \sigma^2/(1-\phi_1^2)$.

More generally, the AR(p) process is defined as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t . \tag{1.3}$$

Again, the autocorrelation function can be found by multiplying (1.3) by X_{t-k}, taking the expected value and dividing by γ_0, thus producing the **Yule-Walker equations**

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}, \quad k = 1, 2, \ldots$$

These are linear recurrence relations, with general solution of the form

$$\rho_k = C_1 \omega_1^{|k|} + \cdots + C_p \omega_p^{|k|} ,$$

where $\omega_1, \ldots, \omega_p$ are the roots of

$$\omega^p - \phi_1 \omega^{p-1} - \phi_2 \omega^{p-2} - \cdots - \phi_p = 0$$

and $C_1, \ldots, C_p$ are determined by $\rho_0 = 1$ and the equations for $k = 1, \ldots, p-1$. It is natural to require $\gamma_k \to 0$ as $k \to \infty$, in which case the roots must lie inside the unit circle, that is, $|\omega_i| < 1$. Thus there is a restriction on the values of $\phi_1, \ldots, \phi_p$ that can be chosen.

1.5 Moving average processes

The **moving average process** of order q is denoted MA(q) and defined by

$$X_t = \sum_{s=0}^{q} \theta_s \epsilon_{t-s} \tag{1.4}$$

where $\theta_1, \ldots, \theta_q$ are fixed constants, $\theta_0 = 1$, and $\{\epsilon_t\}$ is a sequence of independent (or uncorrelated) random variables with mean 0 and variance σ^2.

It is clear from the definition that this is second order stationary and that

$$\gamma_k = \begin{cases} 0, & |k| > q \\ \sigma^2 \sum_{s=0}^{q-|k|} \theta_s \theta_{s+k}, & |k| \le q \end{cases}$$

We remark that two moving average processes can have the same autocorrelation function. For example,

$$X_t = \epsilon_t + \theta \epsilon_{t-1} \quad \text{and} \quad X_t = \epsilon_t + (1/\theta)\epsilon_{t-1}$$

both have $\rho_1 = \theta/(1+\theta^2)$, $\rho_k = 0$, $|k| > 1$. However, the first gives

$$\epsilon_t = X_t - \theta \epsilon_{t-1} = X_t - \theta(X_{t-1} - \theta \epsilon_{t-2}) = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \cdots$$

This is only valid for $|\theta| < 1$, a so-called **invertible process**. No two invertible processes have the same autocorrelation function.

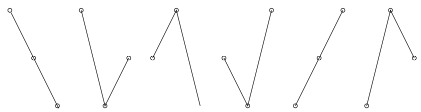
1.6 White noise

The sequence $\{\epsilon_t\}$, consisting of independent (or uncorrelated) random variables with mean 0 and variance σ^2 is called **white noise** (for reasons that will become clear later.) It is a second order stationary series with $\gamma_0 = \sigma^2$ and $\gamma_k = 0$, $k \neq 0$.

1.7 The turning point test

We may wish to test whether a series can be considered to be white noise, or whether a more complicated model is required. In later chapters we shall consider various ways to do this, for example, we might estimate the autocovariance function, say $\{\hat{\gamma}_k\}$, and observe whether or not $\hat{\gamma}_k$ is near zero for all $k > 0$.

However, a very simple diagnostic is the **turning point test**, which examines a series $\{X_t\}$ to test whether it is purely random. The idea is that if $\{X_t\}$ is purely random then three successive values are equally likely to occur in any of the six possible orders.



In four cases there is a turning point in the middle. Thus in a series of n points we might expect $(2/3)(n-2)$ turning points.

In fact, it can be shown that for large n, the number of turning points should be distributed as about $N(2n/3, 8n/45)$. We reject (at the 5% level) the hypothesis that the series is unsystematic if the number of turning points lies outside the range $2n/3 \pm 1.96\sqrt{8n/45}$.