Symmetric Rendezvous Search Games

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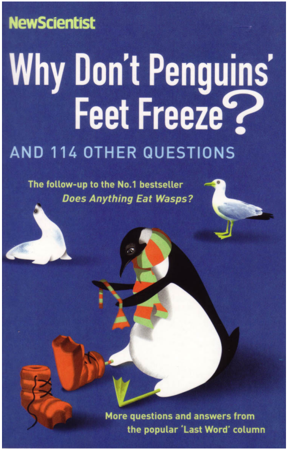
Aisle miles (2006)

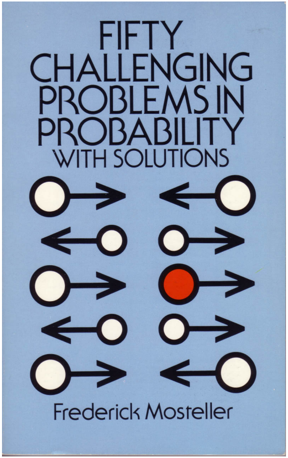


Two people lose each other while wandering through the aisles of a large supermarket.

One person wishes to find the other.

Should that person stop moving and remain in a single visible site while the other person continues to move through the aisles? Or would an encounter or sighting occur sooner if both were moving through the aisles?

Quo vadis? (Mosteller, 1965)

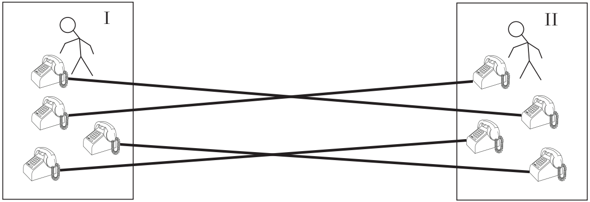


Two strangers who have a private recognition signal agree to meet on a certain Thursday at 12 noon in New York City, a town familiar to neither to discuss an important business deal, but later they discover that they have not chosen a meeting place, and neither can reach the other because both have embarked on trips. If they try nevertheless to meet, where should they go?

Telephone coordination game (Alpern, 1976)

In each of two rooms there is a player and n telephones.

Phones are connected pairwise in some unknown fashion.



At attempts $1, 2, \dots$, the players pick up phones and say "hello".

Their common aim is to minimize the expected number of attempts until they hear one another.

Symmetric rendezvous search on n locations

Assumptions

1. Two players are randomly placed at two distinct of n locations.
2. There is no commonly held labelling of the locations.
3. At each of steps, $1, 2, \dots$, each player visits one of the locations.
4. The players adopt identical (randomizing) strategies.

What should their common strategy be if they wish to meet in the least expected number of steps?

Some possible strategies

Move-at-random If at each discrete step $1, 2, \ldots$ each player were to locate himself at a randomly chosen location, then the expected time to meet would be n. E.g.,

$$ET = 1 + \frac{n-1}{n} ET \implies ET = n\,.$$

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Wait-for-mommy Suppose the players could break symmetry (or had some prior agreement). Now it is best for one player to remain stationary while the other tours all other locations in random order. They will meet (on average) half way through the tour. So

$$ET = \tfrac{1}{n-1}\big(1 + 2 + \cdots + (n-1)\big) = \tfrac{1}{2}n\,.$$

Wait-for-mommy

E.J. Anderson and R.R. Weber. The rendezvous problem on discrete locations. *J. Appl. Prob.* 27, 839-851, 1990.

Theorem 1 *In the asymmetric rendezvous search game on n locations the optimal strategy is wait-for-mommy.*

The Anderson-Weber strategy

Motivated by the optimality of *wait-for-mommy* in the asymmetric case, Anderson and Weber (1990) proposed the following strategy:

AW: If rendezvous has not occurred within the first $(n-1)j$ steps then in the next $n-1$ steps each player should either stay at his initial location or tour the other $n-1$ locations in random order, with probabilities p and $1-p$, respectively, where p is to be chosen optimally.

Facts about **AW**

$n = 2$:

AW with $p = 1/2$ is optimal and equivalent to move-at-random.

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AW with $p = 1/2$ is optimal and equivalent to move-at-random.

As $n \to \infty$:

AW (with $p \to 0.24749$) achieves a meeting time of $\approx 0.8289n$ (which is better than move-at-random).

Anderson-Weber strategy on 3 locations

On 3 locations, **AW** specifies that in each block of two consecutive steps, each player should, independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities $p = \frac{1}{3}$ and $1 - p = \frac{2}{3}$.

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R.R. Weber, Optimal symmetric rendezvous search on three locations, *Math Oper Res.*, 37(1): 111–122, 2012.

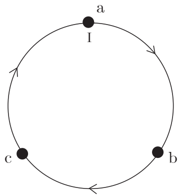
Theorem 3 *On 3 locations,* **AW** *minimizes ET.*

Corollary. *The minimal expected meeting time is $w = \frac{5}{2}$.*

AW gives $ET = \frac{5}{2}$, whereas *move-at-random* gives $ET = 3$.

Formulation of the problem

Suppose the three locations are arranged around a circle.

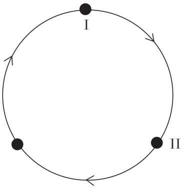


Each player calls his home location 'a', chooses a 'clockwise' direction and labels locations clockwise of home as 'b' and 'c'.

A sequence of a player's moves can now be described.
E.g., a player's first 6 moves might be '$ababbc$'.

Make the problem easier by providing the players with a common notion of clockwise. (We'll see this does not actually help.)

Player II starts one position clockwise of Player I.



$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Matrix B_1 has '1' if after the first step they do not meet, and '0' if they do.

Rows of B_1 correspond to I playing a, b or c.

Columns of B_1 correspond to II playing a, b or c.

The minimum of $P(T > 2)$

The indicator matrix for not meeting within 2 steps is

$$B_2 := B_1 \otimes B_1 = \begin{pmatrix} B_1 & B_1 & 0 \\ 0 & B_1 & B_1 \\ B_1 & 0 & B_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Rows 1–9 (and columns 1–9) correspond respectively to Player I (or II) playing patterns of moves over the first two steps of $aa, ab, ac, ba, bb, bc, ca, cb, cc$.

$ET = \sum_{k=0}^{\infty} P(T > k)$.

AW minimizes $P(T > 2)$

Let $\bar{B}_2 = \frac{1}{2}(B_2 + B_2^\top)$ (to account for II starting either one or two locations clockwise of I).

$$P(T > 2) = p^\top \bar{B}_2 p = p^\top \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} p$$

is to be minimized over probability vectors p.

Minimizer is $p^\top = \frac{1}{3}(1, 0, 0, 0, 0, 1, 0, 1, 0)$, where '$aa$', '$bc$' and '$cb$' are to be chosen equally likely, (which is **AW**).

Another minimizer is $p^\top = (0, 1, 0, 1, 0, 0, 0, 0, 1)$, where '$ab$', '$ba$' and '$cc$' are to be chosen equally likely.

A quadratic programming problem

To prove that **AW** minimizes $p^\top \bar{B}_2 p$ we must solve a difficult quadratic programming problem.

The difficulty arises because $\bar{B}_2$ is not positive semidefinite.
It's eigenvalues are $\{4, 1, 1, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$.
This means that there can be local minima to $p^\top \bar{B}_2 p$.

E.g., $p = \frac{1}{9}(1, 1, 1, 1, 1, 1, 1, 1, 1)$, is a local minimum; but $p^\top \bar{B}_2 p = \frac{4}{9}$. This is not a global minimum.

In general, if a matrix C is not positive semidefinite, the following problem is NP-hard:

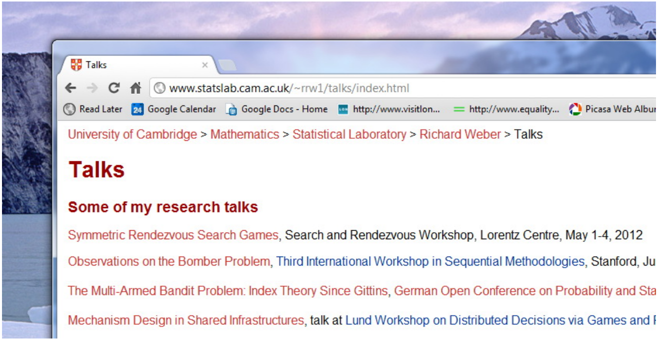
$$\text{minimize } \; p^\top C p \; : \; p \geq 0 \,, \; 1^\top p = 1 \,.$$

Details of proof in slides at the end of this talk

We actually prove that, for all k, **AW** minimizes

$$E[\min\{T, k+1\}] = \sum_{j=0}^{k} P(T > j) = p^\top M_k p.$$

`www.statslab.cam.ac.uk/~rrw1/talks`



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$$ET = \frac{1}{12}\left(15 + \sqrt{681}\right) \approx 3.42466\,.$$

But there is a better strategy with ET less by 0.00014668.

The optimal strategy for 4 locations is unknown.

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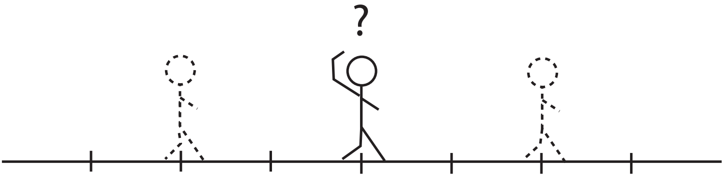
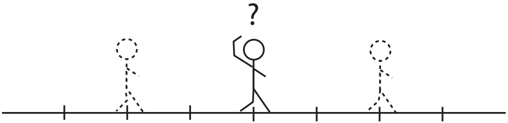
Conjecture: **AW** is asymptotically optimal, in the sense that one can do no better than $ET \sim 0.8289n$.

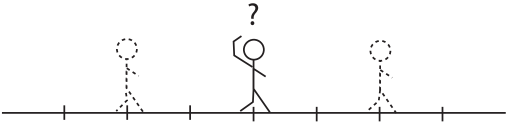
Symmetric rendezvous search on the line

Two players are placed 2 units apart on a line, randomly facing left or right. At each step each player must either move one unit forward or backwards. Each player knows that the other player is equally likely to be in front or behind him, and equally likely to be facing either way. How can they meet in the least expected time?





$4.1820 \leq w \leq 4.2574$ (Improve these bounds?)



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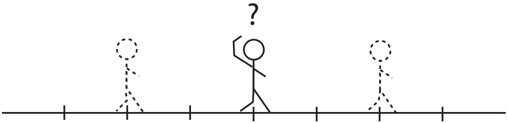

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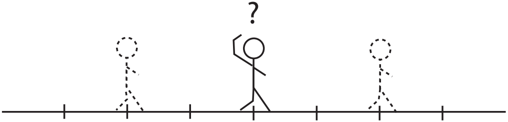

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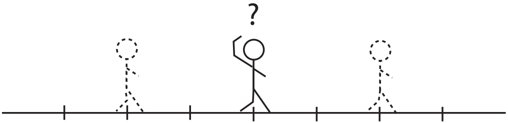
We have seen that on 3 locations it is no help for players to be given a common notion of clockwise. Similarly, here:



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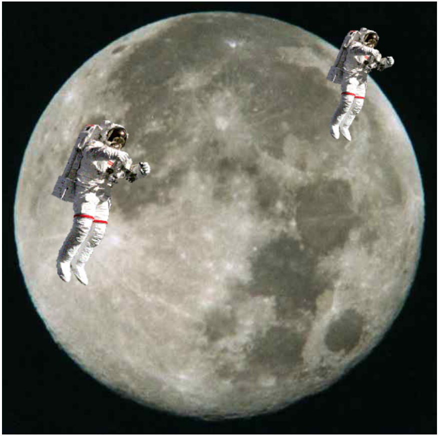
We have seen that on 3 locations it is no help for players to be given a common notion of clockwise. Similarly, here:

Conjecture: it does not to help if players are told that they are initially faced the same way.

Symmetric rendezvous search in other spaces

Alpern (1976) has also proposed the following problem.



Two astronauts land at random spots on a planet (which is assumed to be a uniform sphere, without any known distinguishing marks or directions) How should they move so as to be within 1 kilometre of one another in the least expected time?

Appendix

Proof that **AW** is optimal on 3 locations

A method for finding lower bounds

Suppose we are trying to minimize $p^\top C p$, but C is not positive semidefinite.

We can obtain a lower bound on the solution as follows.

$$\begin{aligned}
&\min\{p^\top C p \,:\, p \geq 0\,,\ 1^\top p = 1\} \\
&\qquad = \min\{\mathrm{trace}(Cpp^\top) \,:\, p \geq 0\,,\ 1^\top p = 1\} \\
&\qquad \geq \min\{\mathrm{trace}(CX) \,:\, X \succeq 0\,,\ X \geq 0\,,\ \mathrm{trace}(JX) = 1\}\,,
\end{aligned}$$

where $J = 11^\top$ is a matrix of all 1s.

This is by using the fact that if p satisfies the l.h.s. constraints, then $X = pp^\top$ satisfies the r.h.s. constraints.

Semidefinite programming problems

'linear programming for the 21st century'.

Given symmetric matrices $C, A_1, \dots, A_m$, consider the problem

$$\text{minimize}\,\{\text{trace}(CX)$$
$$:\ X \succeq 0\,,\ X \geq 0\,,\ \text{trace}(A_i X) = b_i,\ i = 1, \dots, m\}\,.$$

This is a *Semidefinite Programming Problem* (SDP).

The minimization is over the components of X.
This can mean lots of decision variables.
If X is $j \times j$ and symmetric, then there are $j(j-1)/2$ variables.

SDPs can be solved to any degree of numerical accuracy using interior point algorithms (e.g., using Matlab and sedumi).

A lower bound on $p^\top \bar{B}_2 p$

As a relaxation of the quadratic program:

$$\text{minimize}\,\{p^\top \bar{B}_2 p \,:\, p \geq 0\,,\; 1^\top p = 1\}\,,$$

we consider the SDP:

$$\text{minimize}\,\{\text{trace}(\bar{B}_2 X) \,:\, X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(J_2 X) = 1\}\,,$$

where J_2 is the 9×9 matrix of 1s. There are 36 decision variables. We find that the minimum value is $1/3$.

But $p^\top \bar{B}_2 p = 1/3$ for $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$.

So we may conclude that $1/3$ is the minimal value of $p^\top \bar{B}_2 p$.

Lower bounds on $E[\min\{T, k+1\}]$

Let w_k be the minimal possible value of the 'expected $k-$truncated rendezvous time',

$$E[\min\{T, k+1\}] = \sum_{j=0}^{k} P(T > j) = p^\top M_k p\,,$$

where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k\,.$$

To find a lower bound on w_k we consider the SDP:

$$\text{minimize}\,\{\text{trace}(\bar{M}_k X)\;:\; X \succeq 0\,,\; X \geq 0\,,\; \text{trace}(XJ_k) = 1\}\,.$$

Lower bounds on w_k

Solving SDPs, we get

lower bounds when players have a common clockwise:

k	1	2	3	4
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$

lower bounds when players have no common clockwise:

k	1	2	3	4	5
w_k	$\frac{5}{3}$	2	$\frac{20}{9}$	$\frac{21}{9}$	$\frac{65}{27}$[‡]

Observations

1. These lower bounds prove that **AW** minimizes $E[\min\{T, k+1\}]$ as far as $k = 4$.
2. But it is computationally infeasible to go much further. The number of decision variables in the SDP is 3240 when $k = 4$. For $k = 5$ it would be 29403.

A conjecture concerning **AW**

$$ET = \sum_{j=0}^{\infty} P(T > j)\,.$$

AW does not minimize every term in this sum. E.g., **AW** gives $P(T > 4) = \frac{1}{9}$, but there is a strategy with $P(T > 4) = \frac{1}{10}$.

w_k is the minimal value of $E[\min\{T, k+1\}] = \sum_{j=0}^{k} P(T > j)$. It is found by minimizing $p^\top M_k p$, where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k\,.$$

Empirically, the lower bounds for w_k are always achieved by **AW** (and are the same whether or not the players have a common notion of clockwise.) This leads us to conjecture the following.

The optimality of **AW** for 3 locations

Theorem 4 *The* **AW** *strategy is optimal for the symmetric rendezvous search game on* 3 *locations, minimizing* $E[\min\{T, k+1\}]$ *to* w_k *for all* $k = 1, 2, \dots$ *, where*

$$w_k = \begin{cases} \frac{5}{2} - \frac{5}{2} 3^{-\frac{k+1}{2}}, & \text{when } k \text{ is odd,} \\ \frac{5}{2} - \frac{3}{2} 3^{-\frac{k}{2}}, & \text{when } k \text{ is even.} \end{cases}$$

Consequently, the minimal achievable value of ET *is* $w = \frac{5}{2}$*.*

$\{w_k\}_0^\infty = \{1, \frac{5}{3}, 2, \frac{20}{9}, \frac{21}{9}, \frac{65}{27}, \dots\}$.

Proof that **AW** is optimal on 3 locations

We begin by describing how we might prove that a given strategy minimizes $E[\min\{T,3\}] = P(T>0) + P(T>1) + P(T>2)$, or equivalently, that a given p minimizes $p^\top \bar{M}_2 p$.

1. Suppose we are trying to minimize $p^\top \bar{M}_2 p$, but $\bar{M}_2$ is not positive semidefinite.
2. Suppose we can find a matrix H_2, which is positive semidefinite and such that $M_2 \geq H_2$.
3. Suppose we can minimize $p^\top \bar{H}_2 p$. This provides a lower bound on the minimum of $p^\top \bar{M}_2 p$.
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The minimum of $E[\min\{T,3\}]$

We can take $p^\top = \frac{1}{3}(1,0,0,0,0,1,0,1,0)$ and

$$M_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}$$

$$\geq H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

Eigenvalues of $\bar{M}_2$ are $\{19, \frac{5}{2}, \frac{5}{2}, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$, so it is not positive semidefinite.
Eigenvalues of $\bar{H}_2$ are $\{18, 3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0\}$ so $\bar{H}_2 \succeq 0$. Here

$$\bar{H}_2 p = \begin{pmatrix} 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} & \frac{3}{2} & 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{3}{2} & 2 \\ 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 3 & \frac{5}{2} & \frac{5}{2} \\ \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{5}{2} & 3 & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & 2 & \frac{3}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Thus p satisfies a Kuhn-Tucker condition for there to be a local minimum of $p^\top \bar{H}_2 p = 2$.
Since $\bar{H}_2 \succeq 0$, a local minimum is also a global minimum.
So $w_2 = 2$. This is achieved by **AW**.

Minimizing $E[\min\{T, k+1\}]$

Similarly, consider the problem of minimizing $E[\min\{T, k+1\}]$.

This is equivalent to minimizing $p^\top \bar{M}_k p$, where

$$M_k = J_k + B_1 \otimes J_{k-1} + \cdots + B_k \,.$$

As we did with H_2 for M_2, we look for H_k, such that $H_k \leq M_k$ and $\bar{H}_k \succeq 0$. This is a semidefinite programming problem

$$\text{maximize}\{\text{trace}(J_k H_k) \,:\, H_k \leq M_k,\; \bar{H}_k \succeq 0\}\,.$$

How can we find H_k?

$$\text{maximize}\{\text{trace}(J_2 H_2) \,:\, H_2 \leq M_2,\; \bar{H}_2 \succeq 0\}\,.$$

How can we find H_k?

$$\text{maximize}\{\text{trace}(J_2 H_2) \;:\; H_2 \leq M_2,\;\; \bar{H}_2 \succeq 0\}\,.$$

$$H_2 = \begin{pmatrix}
3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 \\
1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 \\
2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 \\
0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 \\
0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 \\
1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 \\
2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 \\
2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 \\
2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000
\end{pmatrix}$$

and $\min_p\{p^\top H_2 p\} = 1.9999889.$

How can we find H_k?

$$\text{maximize}\{\text{trace}(J_2H_2)\ :\ H_2 \le M_2,\ \bar{H}_2 \succeq 0\}\,.$$

$$H_2 = \begin{pmatrix} 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 \\ 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 \\ 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 \\ 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 & 2.8005 & 2.8005 & 2.0000 \\ 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 & 2.0000 & 2.8005 & 2.8005 \\ 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 & 2.8005 & 2.0000 & 2.8005 \\ 2.8005 & 2.8005 & 2.0000 & 0.8857 & 1.0000 & 0.8857 & 3.0000 & 2.7951 & 1.8324 \\ 2.0000 & 2.8005 & 2.8005 & 0.8857 & 0.8857 & 1.0000 & 1.8324 & 3.0000 & 2.7951 \\ 2.8005 & 2.0000 & 2.8005 & 1.0000 & 0.8857 & 0.8857 & 2.7951 & 1.8324 & 3.0000 \end{pmatrix}$$

and $\min_p\{p^\top H_2 p\} = 1.9999889$. But $\min_p\{p^\top H_2 p\} = 2$ using

$$H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

How to construct H_k

Let us search for H_k of a special form. For $i = 0, \ldots, 3^k - 1$ we write $i_{\text{base}\,3} = i_1 \cdots i_k$ (keeping k digits, including leading 0s); so $i_1, \ldots, i_k \in \{0, 1, 2\}$. Define

$$P_i = P_{i_1 \cdots i_k} = P_1^{i_1} \otimes \cdots \otimes P_1^{i_k} \, ,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

Observe that $M_k = \sum_i m_k(i) P_i$, where m_k is the first row of M_k. This motivates seeking H_k of the form

$$H_k = \sum_{i=0}^{3^k - 1} x_k(i) P_i \, .$$

Concluding steps of the proof

We want

1. $M_k = \sum_i m_k(i) P_i \geq H_k = \sum_i x_k(i) P_i$.
2. $\bar{H}_k \succeq 0$.

Since $P_0, \ldots, P_{3^k-1}$ commute they have common eigenvectors. Let $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$, a cube root of 1. Let $V_k = U_k + iW_k$.

$$V_k = V_1 \otimes V_{k-1}\,, \quad \text{where } V_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Columns of V_k are eigenvectors of the P_i and also of M_k. Columns of U_k are eigenvectors of the $\bar{P}_i$ and also of $\bar{M}_k$.

Our SDP becomes equivalent to a LP, with constraints 1. $m_k \geq x_k$ and 2. $U_k x_k \geq 0$.

We show that we may take $H_k = \sum_i x_k(i) P_i$, where

$$x_1 = (2,2,1)^\top \quad x_2 = (3,3,2,3,3,2,1,1,0)^\top$$

and choose a_k so that for $k \geq 3$,

$$\begin{aligned} x_k = & 1_k + (1,0,0)^\top \otimes x_{k-1} \\ & + (0,1,0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3} \,. \end{aligned}$$

Here a_k is chosen maximally such that $U_k x_k \geq 0$ and $m_k \geq x_k$.

All rows of H_k have the same sum, and so $p^\top H_k p$ is minimized by $p = (1/3^k) 1_k$, and the minimum value is $p^\top H_k p = 1_k^\top x_k / 3^k$.

So the theorem is true provided $1^\top x_k = 3^k w_k$.

$1^\top x_k = 3^k w_k$ iff we can take

$$a_k = \begin{cases} 3 - \dfrac{1}{3^{(k-3)/2}}, & \text{when } k \text{ is odd,} \\ 3 - \dfrac{2}{3^{(k-2)/2}}, & \text{when } k \text{ is even.} \end{cases}$$

Note that a_k increases monotonically in k, from 2 towards 3. As $k \to \infty$ we find $a_k \to 3$ and $1_k^\top x_k / 3^k \to \frac{5}{2}$.

Finally, we prove that with these a_k we have always have

1. $m_k \geq x_k$, (implying $M_k \geq H_k$).
2. $U_k x_k \geq 0$, (implying $\bar{H}_k \succeq 0$).

Both are proved by induction. The first is easy and the second is hard. To prove the second we use the recurrence relation for x_k to find recurrences relations for components of the vectors $U_k x_k$, and then show that all components are nonnegative. ■

Proof that **AW** is not optimal on 4 locations

Anderson-Weber strategy on 4 locations

On 4 locations the expected rendezvous time under **AW** satisfies

$$
\begin{aligned}
ET &= p^2(3+ET) + 2p(1-p)2 + (1-p)^2\left(\tfrac{1}{2}\tfrac{16}{9} + \tfrac{1}{2}(3+ET)\right) \\
&= \frac{43-14p+25p^2}{9\left(1+2p-3p^2\right)}\,.
\end{aligned}
$$

The minimum of ET is achieved by taking

$$
p = \tfrac{1}{4}\left(3\sqrt{681}-77\right) \approx 0.321983\,,
$$

which lead to

$$
ET = \tfrac{1}{12}\left(15+\sqrt{681}\right) \approx 3.42466\,.
$$

Suppose location 1 (2) is the home location of player I (II).
Each player independently labels his non-home locations as a, b, c.
A tour of non-home locations is one of abc, acb, bac, bca, cab, cba.

Suppose location 1 (2) is the home location of player I (II).
Each player independently labels his non-home locations as a, b, c.
A tour of non-home locations is one of abc, acb, bac, bca, cab, cba.
If I has $(a, b, c) = (2, 3, 4)$ and II has $(a, b, c) = (1, 3, 4)$ we find

$$B = \begin{pmatrix} 2 & \text{X} & 3 & \text{X} & \text{X} & 2 \\ \text{X} & 2 & \text{X} & 2 & 3 & \text{X} \\ 3 & \text{X} & 1 & 1 & \text{X} & \text{X} \\ \text{X} & 2 & 1 & 1 & \text{X} & \text{X} \\ \text{X} & 3 & \text{X} & \text{X} & 1 & 1 \\ 2 & \text{X} & \text{X} & \text{X} & 1 & 1 \end{pmatrix}$$

Rows and columns to correspond to abc, acb, bac, bca, cab, cba.
A number shows the step at which players meet.
X indicates that they do not meet.

Suppose location 1 (2) is the home location of player I (II).
Each player independently labels his non-home locations as a, b, c.
A tour of non-home locations is one of abc, acb, bac, bca, cab, cba.
If I has $(a, b, c) = (2, 3, 4)$ and II has $(a, b, c) = (1, 3, 4)$ we find

$$B = \begin{pmatrix} 2 & \text{X} & 3 & \text{X} & \text{X} & 2 \\ \text{X} & 2 & \text{X} & 2 & 3 & \text{X} \\ 3 & \text{X} & 1 & 1 & \text{X} & \text{X} \\ \text{X} & 2 & 1 & 1 & \text{X} & \text{X} \\ \text{X} & 3 & \text{X} & \text{X} & 1 & 1 \\ 2 & \text{X} & \text{X} & \text{X} & 1 & 1 \end{pmatrix}$$

Rows and columns to correspond to abc, acb, bac, bca, cab, cba.
A number shows the step at which players meet.
X indicates that they do not meet.
There are 36 such matrices, over which we must average, for each possible pair of assignments by players I and II, of $(2, 3, 4)$ and $(1, 3, 4)$, respectively, to (a, b, c).

$$\begin{pmatrix} 2 & X & 3 & X & X & 2 \\ X & 2 & X & 2 & 3 & X \\ 3 & X & 1 & 1 & X & X \\ X & 2 & 1 & 1 & X & X \\ X & 3 & X & X & 1 & 1 \\ 2 & X & X & X & 1 & 1 \end{pmatrix} \begin{pmatrix} X & 2 & X & 2 & 3 & X \\ 2 & X & 3 & X & X & 2 \\ X & 3 & X & X & 1 & 1 \\ 2 & X & X & X & 1 & 1 \\ 3 & X & 1 & 1 & X & X \\ X & 2 & 1 & 1 & X & X \end{pmatrix} \begin{pmatrix} 3 & X & 2 & X & 2 & X \\ X & 2 & X & 2 & X & 3 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \end{pmatrix} \begin{pmatrix} X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \end{pmatrix} \begin{pmatrix} 2 & X & X & 3 & X & 2 \\ X & 3 & 2 & X & 2 & X \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \end{pmatrix}$$

$$\begin{pmatrix} X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \end{pmatrix} \begin{pmatrix} X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \\ 3 & X & 2 & X & 2 & X \\ X & 2 & X & 2 & X & 3 \\ X & X & 2 & X & 1 & 1 \\ X & X & X & 3 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ X & X & 1 & 1 & 2 & X \\ X & X & 1 & 1 & X & 3 \end{pmatrix} \begin{pmatrix} X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ 1 & 1 & X & 2 & X & X \\ 1 & 1 & 3 & X & X & X \end{pmatrix} \begin{pmatrix} X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \\ 2 & X & X & 3 & X & 2 \\ X & 3 & 2 & X & 2 & X \\ 1 & 1 & X & X & X & 2 \\ 1 & 1 & X & X & 3 & X \end{pmatrix}$$

A new search game on 6 locations

When a player makes a tour in **AW** he chooses it at random. Might something else be better?

Consider a new game, in which at each new step (of 3 old steps) each player makes a tour of his non-home locations.

Let AAB denote three successive tours: the first tour is chosen at random, the second is chosen to be the same as the first, and the third is chosen randomly from amongst the 5 not yet tried.

If successive tours are chosen at random,

$$ET = 1 + \frac{1}{2}ET$$

so $ET = 2$.

The optimal 2–Markov policy

Over two steps possible strategies are AA and AB. We find a non-meet matrix of

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{5} & \frac{13}{50} \end{pmatrix}$$

So

$$ET = p^\top \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{5} & \frac{13}{50} \end{pmatrix} ET \right) p$$

and $\begin{pmatrix} & \end{pmatrix} \succ 0$. This is minimized by $p^\top = (1/6, 5/6)$, so in fact it is optimal to choose tours at random.

The optimal 3–Markov policy

Now possible strategies over 3 steps are AAA, AAB, ABA, ABB, ABC. The not-meeting matrix is

$$P_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{5} & \frac{13}{50} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} \\ \frac{1}{5} & \frac{2}{25} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} \\ \frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} \\ \frac{1}{20} & \frac{11}{100} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} \end{pmatrix}$$

We find $P_3 \succeq 0$. Again, it turns out that choosing tours at random is optimal, $p^\top = (1, 5, 5, 5, 20)/6^2$.

A 4–Markov policy better than **AW**

Over 4 steps there are 15 possible strategies: $AAAA$, $AAAB$, $AABA$, $AABB$, $AABC$, $ABAA$, $ABAB$, $ABAC$, $ABBA$, $ABBB$, $ABBC$, $ABCA$, $ABCB$, $ABCC$, $ABCD$.

$P_4 =$

$$\begin{pmatrix}
\frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & 0 \\
\frac{1}{5} & \frac{13}{50} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{11}{100} & \frac{11}{100} & \frac{1}{50} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{75} & \frac{2}{25} & \frac{1}{30} & \frac{1}{30} & \frac{1}{30} & \frac{11}{100} & \frac{23}{450} \\
\frac{1}{20} & \frac{11}{100} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{1}{30} & \frac{1}{50} & \frac{7}{150} & \frac{7}{150} & \frac{7}{150} & \frac{1}{20} & \frac{14}{225} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{11}{100} & \frac{1}{50} & \frac{11}{100} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{2}{75} & \frac{2}{25} & \frac{1}{30} & \frac{1}{30} & \frac{11}{100} & \frac{1}{30} & \frac{23}{450} \\
\frac{1}{20} & \frac{11}{100} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{30} & \frac{1}{50} & \frac{7}{150} & \frac{7}{150} & \frac{1}{20} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{11}{100} & \frac{1}{30} & \frac{1}{30} & \frac{23}{450} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{1}{50} & \frac{11}{100} & \frac{11}{100} & \frac{3}{100} \\
\frac{1}{20} & \frac{11}{100} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{20} & \frac{7}{150} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{1}{50} & \frac{1}{20} & \frac{7}{50} & \frac{7}{150} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{1}{50} & \frac{11}{100} & \frac{1}{20} & \frac{1}{30} & \frac{11}{100} & \frac{7}{150} & \frac{7}{150} & \frac{7}{50} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{1}{50} & \frac{11}{100} & \frac{1}{20} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{1}{30} & \frac{11}{100} & \frac{7}{150} & \frac{7}{150} & \frac{7}{150} & \frac{7}{50} & \frac{14}{225} \\
0 & \frac{3}{100} & \frac{3}{100} & \frac{23}{450} & \frac{14}{225} & \frac{3}{100} & \frac{23}{450} & \frac{14}{225} & \frac{23}{450} & \frac{3}{100} & \frac{14}{225} & \frac{14}{225} & \frac{14}{225} & \frac{14}{225} & \frac{7}{90}
\end{pmatrix}$$

P_4 has a negative eigenvalue. Choosing tours at random is

$$p^\top = \frac{1}{6^3}(1, 5, 5, 5, 20, 5, 5, 20, 5, 5, 20, 20, 20, 20, 60)\,.$$

and this gives $ET = 2$. However, using

$$p^\top = \frac{1}{12}(0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 8)$$

we get $ET = 2 - \frac{23}{16200}$.

Players do $AAAB$, $AABA$, $ABAA$, $ABBB$ each with probability $1/12$, and $ABCD$ with probability $2/3$.

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This is like **AW**. With probability $p = 1/3$ a player does his home tour A and one other tour B. With probability $p = 2/3$ he tours 3 other non-home tours B, C, D.

A strategy better than **AW** for 4 locations

Consider a 12-Markov strategy consisting of four 3-steps. In each 3-step a player remains home with probability p, or tours his non-home locations with probability $1-p$. It is **AW**, except that when a player makes tours he does so as previously described. Any 1st and 2nd tours are made at random, but then 3rd and 4th tours are made such that $AAAB$, $AABA$, $ABAA$, $ABBB$ have probabilities $1/12$, and $ABCD$ has probability $2/3$.
There are 1585 possible paths of nonzero probability. Careful computation finds $ET =$

$$\frac{-227773p^8 + 582884p^7 - 1329319p^6 + 1737938p^5 - 1941235p^4 + 1420688p^3 - 998569p^2 + 389834p - 217648}{3\left(82001p^8 - 218608p^7 + 327728p^6 - 315256p^5 + 215870p^4 - 104656p^3 + 36128p^2 - 8008p - 15199\right)}$$

For $p = (1/4)(3\sqrt{681} - 77)$ (same as **AW**) this gives ET less than **AW** by 0.00014668.