# Bargaining, Arbitration and Voting Games:

## Some Mathematics of Fair Division

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## The Disputed Garment Problem

The Babylonian Talmud is the compilation of ancient law and tradition set down during the first five centuries A.D. which serves as the basis of Jewish religious, criminal and civil law. One problem discussed in the Talmud is the so-called **disputed garment problem**.

> "Two hold a garment; one claims it all, the other claims half. Then one is awarded $\frac{3}{4}$ and the other $\frac{1}{4}$."

The idea here is that half of the garment is not in disputer and can be awarded to the one who claims the whole garment. The other half of the garment is in dispute and should be split equally.

Thus one gets $\frac{1}{2} + \frac{1}{4}$ and the other gets $\frac{1}{4}$.

# The Marriage Contract Problem

Another problem discussed in the Talmud is the so-called **marriage contract problem**.

A man has three wives whose marriage contracts specify that in the case of this death they receive $\mathbf{100}$, $\mathbf{200}$ and $\mathbf{300}$ respectively. The Talmud gives apparently contradictory recommendations.

| | Debt | | |
|---|---|---|---|
| Estate | $\mathbf{100}$ | $\mathbf{200}$ | $\mathbf{300}$ |
| 100 | $\mathbf{33\frac{1}{3}}$ | $\mathbf{33\frac{1}{3}}$ | $\mathbf{33\frac{1}{3}}$ |
| 200 | $\mathbf{50}$ | $\mathbf{75}$ | $\mathbf{75}$ |
| 300 | $\mathbf{50}$ | $\mathbf{100}$ | $\mathbf{150}$ |

Thus when the man dies leaving an estate of only $\mathbf{100}$, the Talmud recommends equal division. However, if the estate is worth $\mathbf{300}$ it recommends proportional division ($\mathbf{50}$,$\mathbf{100}$,$\mathbf{150}$), while for an estate of $\mathbf{200}$, its recommendation of ($\mathbf{50}$,$\mathbf{75}$,$\mathbf{75}$) is a complete mystery.



# The Bankruptcy Game

Two creditors, 1 and 2, have valid claims for $\pounds 30$ million and $\pounds 70$ million against a bankrupt company. But the company only has $\pounds 60$ million.

The players have to reach an agreement about how to divide the money between them, i.e., to choose $a_1, a_2$, such that

creditor 1 gets $a_1$ and creditor 2 gets $a_2$;

and $a_1 + a_2 \leq 60$.

Both Players are equally powerful, i.e., have equally good lawyers, etc.

Once all the arguments have been made and ‘the dust has settled’ how much money do you think each will get?

What would be a ‘fair’ division of the money?

# Some Lawyer's Arguments

Creditors 1 and 2 have valid claims for $30$ and $70$. But there is only $60$ to divide.

(a) Equal division: $(30, 30)$.

(b) A $30 : 70$ split, proportional to the debts: $(18, 42)$.

(c) A $30 : 60$ split, proportional to the amounts they could get if the other was not a creditor: $(20, 40)$.

(d) Suppose we follow the Talmud, and use the disputed garment principle. This says that Creditor 2 should certainly be awarded at least $30$, since this is what would be left for him if he first paid Creditor 1's entire claim,

$$30 = 60 - 30 \text{ (Creditor 1's entire claim)}.$$

This leaves $30$ in dispute, and it is fair to split that equally, giving $(15, 45)$.

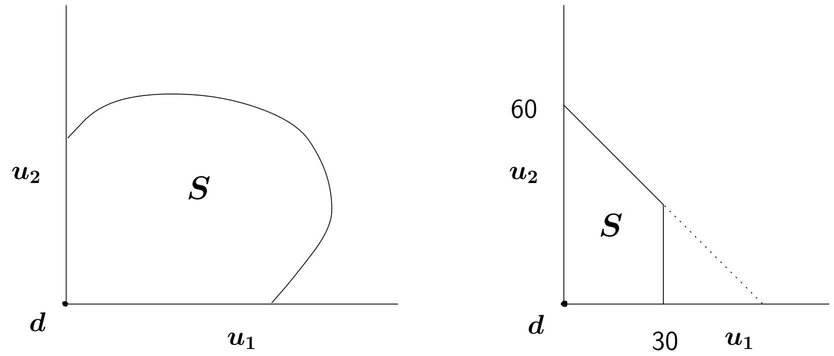
With this division each gets less than he would if he were the only creditor, i.e.,

$$(30, 60) - (15, 45) = (15, 15).$$



# Nash's Bargaining Game

We can represent the bargaining game in the following picture.




Two players attempt to agree on some point $u = (u_1, u_2)$, chosen in the set $S$.

If they agree on $u = (u_1, u_2)$ their 'happinesses' are $u_1$ and $u_2$ respectively.

If they cannot agree they get nothing ($d = 0$ is the 'disagreement point'.)

## Some Reasonable Requirements

Let us think of some reasonable ‘axioms’ upon which the creditors could agree.

1. **Efficiency** The whole $60$ should be split between them, i.e., no money should be thrown away.
2. **Symmetry**. If their claims are exactly the same then it would be fair to split the money in half.
3. **Independence of Irrelevant Alternatives**.

   Suppose you and I are deciding upon a pizza to order and share. We decide on a pepperoni pizza, with no anchovies. Suppose that, just as we are about to order, the waiter tells us that the restaurant is out of anchovies. Knowing this, it would now be silly to decide to switch to having a mushroom pizza.

   The fact that anchovies are not available is irrelevant, since we did not want them anyway.



## Nash Solution of the Bargaining Game

**Theorem 1** *There is one and only one way to satisfy the axioms of efficiency, symmetry and independence of irrelevant alternatives.*

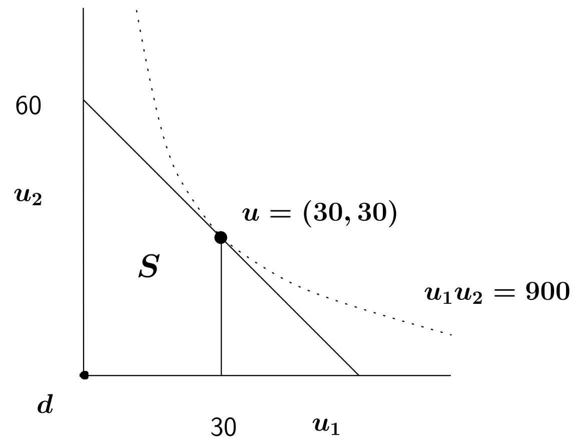
*It is to choose the point in $S$ which maximizes*

$$u_1 u_2.$$

### Solution of the bankruptcy problem

Suppose creditors 1 and 2 have valid claims for $30$ and $70$. But there is only $60$ to share.

The Nash bargaining solution is $u = (30, 30)$.



# Rubenstein's Analysis

Suppose Player 1 makes an offer $(u_1, u_2)$. Player 2 can either accept $u_2$, or make a counter-offer of $(v_1, v_2)$. Player 1 can then accept $v_1$ or make his own counter-offer, and so on.

Suppose the period of time between offers is $\delta$ and rewards are discounted at rate $\alpha$. The setup here is stationary, so in optimal play each player will always make the same offer whenever it is his turn.

So Player 1 must be indifferent between $u_1$ and $v_1 e^{-\alpha\delta}$. Similarly Player 2 must be indifferent between $v_2$ and $u_2 e^{-\alpha\delta}$. Hence

$$u_1 = v_1 e^{-\alpha\delta}, \quad v_2 = u_2 e^{-\alpha\delta} \quad \Longrightarrow \quad u_1 u_2 = v_1 v_2.$$

So $u$ and $v$ lie on a curve $u_1 u_2 = v_1 v_2$=constant, and are both on the boundary of $S$. Also $|u - v| \to 0$ as $\delta \to 0$. These imply that $u, v$ tend to the point where $\bar{u}$, where $\bar{u}_1 \bar{u}_2$ is maximized in $S$, i.e., the Nash bargaining point.



# Zeuthen's Analysis

Suppose there is no discounting. Suppose Player 1's last offer was $(u_1, u_2)$ and that Player 2's last offer was $(v_1, v_2)$. Who should make the next concession?

Player 1 can either accept $v_1$ or he can refuse to to budge and insist on his own last offer. If he thinks there is a probability $p$ that Player 2 will accept his last offer, (and if Player 2 does not accept then Player 1 will be left with $0$), then it makes sense to refuse to budge if

$$pu_1 + (1-p)0 \geq v_1, \quad \text{i.e., if} \quad p \geq r_1 := v_1/u_1.$$

Similarly, it makes sense for Player 2 to refuse to budge if he thinks the probability that Player I will accept his offer is at least $r_2 := u_2/v_2$.

It is reasonable that Player 1 should be the one to make the next concession if $r_1 > r_2$, i.e., if

$$v_1 v_2 > u_2 u_1\,.$$

# Zeuthen's solution

We have seen that it makes sense that Player 1 should be the one to make a concession if $r_1 > r_2$, i.e., if

$$v_1 v_2 > u_2 u_1$$

I.e., it is the player with the smaller Nash product who should make a concession.

Thus we may imagine negotiation taking place in rounds of arguing, posturing, bluffing and conceding. The player with the smaller Nash product is the one who should eventually make a concession at each round, conceding enough that the other player now has the smaller Nash product. Assuming that the size of the concessions are bounded below (e.g., the concessions are made in units of at least $\pounds 1$) then negotiation should finish at the Nash bargaining point where $u_2 u_1$ is maximized. This point is unique.



# Objections and counterobjections

Player 1 is said to have an **objection** to $(u_1, u_2)$ if for some other point in $S$, say $(v_1, v_2)$, there is probability $p$ that he can force Player 2 to accept $v$, and probability $1 - p$ that negotiations breakdown. He prefers $v$ because

$$pv_1 + (1-p)0 \geq u_1.$$

Player 2 is said to have a valid **counterobjection** is

$$pu_2 + (1-p)0 \geq v_2.$$

That is, Player 2 prefers to insist on the original point, even at the risk of negotiations breaking down.

Let us define $u$ as a point in $S$ such that every objection has a *valid counterobjection*.

Note that $v = (v_1, v_2)$ is an objection to $u$ if $p = u_1/v_1$. Since we are supposing there is a counterobjection to this, we must have $p \geq v_2/u_2$. Hence

$$p = u_1/v_1 \geq v_2/u_2 \quad \text{or} \quad u_1 u_2 \geq v_1 v_2.$$

Thus $u$ is the point in $S$ maximizing $u_1 u_2$.

# Back to the Talmud

What about the Talmud division of an estate?

A man has three wives whose marriage contracts specify that in the case of this death they receive $100$, $200$ and $300$ respectively. The Talmud gives apparently contradictory recommendations.

| | Debt | | |
|---|---|---|---|
| Estate | $100$ | $200$ | $300$ |
| 100 | $33\frac{1}{3}$ | $33\frac{1}{3}$ | $33\frac{1}{3}$ |
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This particular Mishna has baffled Talmudic scholars for two millennia. In 1985, it was recognised that the Talmud anticipates the modern game theory.

The Talmud's solution is equivalent to the nucelolus of an appropriately defined cooperative game. The nucleolus is defined in terms of objections and counterobjections.



# Consistency

**The consistency principle**. If the division amongst $n$ players gives $i, j$ amounts $a_i$ and $a_j$, then these are the same amounts they would get in the solution to a problem in which $a_i + a_j$ is to be divided between $i$ and $j$.

Recall the disputed garment principle:

> "Two hold a garment; one claims it all, the other claims half. Then one is awarded $\frac{3}{4}$ and the other $\frac{1}{4}$."

**Theorem 2** *The Talmud solution is the unique solution that is consistent with the disputed garment principle.*

In other words, if everyone likes the disputed garment principle, then the Talmud solution avoids the possibility that any two people will disagree about how what they have between them has been split.

| | Debt | | |
|---|---|---|---|
| Estate | $100$ | $200$ | $300$ |
| 100 | $33\frac{1}{3}$ | $33\frac{1}{3}$ | $33\frac{1}{3}$ |
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## Good Aspects of the Nash solution

The Nash bargaining solution also extends to bargaining with $n > 2$ players.

The solution is to maximize $u_1 u_2 \cdots u_n$ over $u \in S$.

This has two good properties:

**Consistency**.

**Proportional fairness**.

Suppose $\bar{u}$ is the Nash solution and $u$ is any other solution. Then

$$\sum_{i=1}^{n} \frac{u_i - \bar{u}_i}{\bar{u}_i} \leq 0.$$

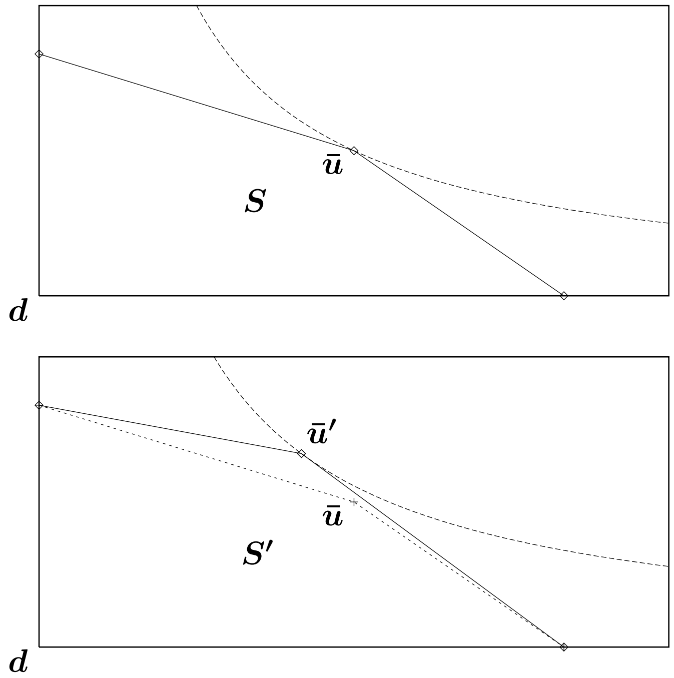
That is, for any move away from the Nash solution the sum of the percentage changes in the utilities is negative.



## A Deficiency of the Nash solution

The Nash bargaining solution does not have the property of **Monotonicity**.

I.e., if $\bar{u}$ and $\bar{u}'$ are the solutions for bargaining sets $S$ and $S'$ respectively, and $S$ is contained in $S'$, then it is not necessarily the case that $\bar{u}' \geq \bar{u}$.



# Arrow's Impossibility Theorem

Consider an election with $n \geq 2$ voters and $m \geq 3$ candidates. Each voter has his own preference rank amongst the candidates. On the basis of these ranks we would like to compute a preference ranking for society, taken as a whole. This preference rank, $\succ$, should satisfy properties of

**Existence**. $\succ$ should be defined for every profile of individual preferences.

**Monotonicity**. If $x \succ y$ and then some individual preferences between $x$ and other candidates are altered in favour of $x$ then we still have $x \succ y$.

**Independence of irrelevant alternatives**. If $x \succ y$ and then some individual preferences between candidates other than $x$ and $y$ are altered, then we still have $x \succ y$.

**Citizen sovereignty** For each pair $x, y$ there is some profile of individual preferences which would give $x \succ y$.

**Non-dictatorship**. There is no individual such that society's preferences are always the same as hers.

**Arrow's Impossibility Theorem**

*There is no way to define a preference ranking for society that satisfies all of the above 5 properties.*



# Coalitions in Provision of Telecommunications Links

The savings in the costs of providing links for Australia, Canada, France, Japan, UK and USA can be defined as

$$v(S) = (\text{cost separate}) - (\text{cost with coalition } S).$$

| subset $S$ | separate | coalition | $v(S)$ | saving (%) |
|---|---|---|---|---|
| J UK USA | 13895 | 11134 | 2761 | 20 |
| A UK USA | 12610 | 10406 | 2204 | 17 |
| F J USA | 6904 | 5609 | 1295 | 19 |
| A F USA | 5600 | 4801 | 799 | 14 |
| C J UK | 3995 | 3199 | 796 | 20 |
| A C UK | 3869 | 3127 | 742 | 19 |
| A J UK USA | 18558 | 13573 | 4985 | 27 |
| F J UK USA | 20248 | 16733 | 3515 | 17 |
| A F J USA | 18847 | 15982 | 2865 | 15 |
| C J UK USA | 15860 | 13044 | 2816 | 18 |
| A F J UK USA | 24990 | 19188 | 5802 | 23 |
| A C J UK USA | 20667 | 15570 | 5097 | 25 |

# Arbitration

Consider a set of $n$ players, $N = \{1, 2, \dots, n\}$. The value they can jointly get from cooperation in some activity is $v(N)$. If a coalition of a subset, $T$, cooperate, they get $v(T)$.

Assume this satisfies for disjoint sets $T$ and $U$,

$$v(T \cup U) \geq v(T) + v(U).$$

E.g., in a bankruptcy in which creditor $i$ is owed $c_i$ and estate is $E$, we could have: $v(T) = \max\{E - \sum_{i \notin T} c_i, 0\}$. This is the amount left after everyone not in $T$ is paid.

The job of an arbitrator is to 'divide the spoils' of the grand coalition, e.g., to make an award $x_1, \dots, x_n$, (called an **imputation**), to players $1, \dots, n$, such that $\sum_{i \in N} x_i = v(N)$, and in a manner to which no can object.

His arbitration decisions are encapsulated in a function $\phi$ which divides $v(N)$ as $x = \big(\phi_1(N), \dots, \phi_n(N)\big)$.

$\phi(\cdot)$ also encapsulates the way the arbitrator would divide $v(T)$ amongst the members of any subset $T \subset N$.



# Objections and Counterobjections

If $\phi_j(N) > \phi_j(N - \{i\})$, then player $i$ might threaten player $j$, "*give me more or I will leave the coalition and you will lose.*"

Player $j$ has a valid counterobjection if he can point out that if he leaves the coalition then $i$ loses just as much.

On the other hand, if $\phi_j(N) < \phi_j(N - \{i\})$, player $j$ might threaten player $i$, "*give me more or I will convince the others to exclude you and those of us who are left will have more to share.*"

Player $i$ has a valid counterobjection if he can point out that if $j$ is excluded those who remain will be better off by exactly the same amount.

Thus if the arbitrator is to make sure that every such objection has a counterobjection, he must ensure

$$\phi_i(N) - \phi_i(N - \{j\}) = \phi_j(N) - \phi_j(N - \{i\}).$$

## Shapley Value

So if each objection has a counterobjection, we require

$$\phi_i(N) - \phi_i(N - \{j\}) = \phi_j(N) - \phi_j(N - \{i\}).$$

There is only one function $\phi(\cdot)$ which does this. It is called the **Shapley value**. Its value for player $i$ is the expected amount he brings to the coalition when the coalition is formed in random order.

In bankruptcy with estate $E$ and creditor $i$ claiming $c_i$, let

$$v(T) = \max\left\{E - \sum_{i \notin T} c_i, 0\right\}.$$

i.e., the amount of money left (if any) once everyone not in $T$ has had his claim paid in full. E.g., $c_1 = 30$, $c_2 = 70$, $E = 60$, gives $v(\{1\}) = 0$, $v(\{2\}) = 30$ and $v(\{1, 2\}) = 60$.

For joining order 1,2: 1 brings $0$, then 2 brings $60$.

For joining order 2,1: 2 brings $30$, then 1 brings $30$.

Therefore

$$\phi_1(N) = \tfrac{1}{2}(0 + 30) = 15,\ \phi_2(N) = \tfrac{1}{2}(60 + 30) = 45.$$



## Sharing the Cost of a Runway

The Shapley value has been used for cost sharing.

Suppose three airplanes share a runway. The planes require 1, 2 and 3 km to land, respectively. So a runway of 3km must be built. How much should each pay?

| | adds cost | | |
|---|---|---|---|
| order | 1 | 2 | 3 |
| 1,2,3 | 1 | 1 | 1 |
| 1,3,2 | 1 | 0 | 2 |
| 2,1,3 | 0 | 2 | 1 |
| 2,3,1 | 0 | 2 | 1 |
| 3,1,2 | 0 | 0 | 3 |
| 3,2,1 | 0 | 0 | 3 |
| Total | 2 | 5 | 11 |

So they should pay for $2/6$, $5/6$ and $11/6$ km, respectively.

# Shapley Value

The intuition behind the Shapley value is that it represents each player's bargaining power in terms of a percentage of the total value created. Bargaining power varies with value contributed. Persons who contribute more receive a higher percentage of the benefits.

The Shapley value is also the only value which satisfies four axioms, namely,

(1) treatment of all players is symmetric,

(2) non-contributors receive nothing,

(3) the division is Pareto efficient, and

(4) for multiple games, the expected value of the sum is the sum of the expected values.

It also accords well with other efficiency concepts such as Nash equilibrium.



# Political Power

The Shapley value has also been used to assess political power.

In 1964 the Board of Supervisor of Nassau County operated by weighted voting. There were six members, with weights of $\{31, 31, 28, 21, 2, 2\}$.

Majority voting operates, so for $T \subseteq \{1, 2, 3, 4, 5, 6\}$, let

$$v(T) = \begin{cases} 1 \text{ if } T \text{ has a total weight } 58 \text{ or more;} \\ 0 \text{ otherwise.} \end{cases}$$

The Shapley values are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\right)$.

Nobody had realised that 3 members were totally without influence.

# The Nucleolus

A final characterization of the Talmud solution is the following.

$x$ is called an **imputation** (a 'division of the spoils') if

$$\sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i.$$

Suppose that for all imputations $y$ and subsets $T \subseteq N$ such that $\sum_{i \in T} y_i > \sum_{i \in T} x_i$ there exists some $U \subseteq N$ such that

$$v(U) - \sum_{i \in U} y_i \geq v(U) - \sum_{i \in U} x_i \geq v(T) - \sum_{i \in T} x_i$$

then $x$ is said to be in the **nucleolus** of the coalitional game. It is a theorem that the nucleolus exists and is a single point.

Suppose we again take

$$v(T) = \max \left\{ E - \sum_{i \notin T} c_i, 0 \right\}$$

i.e., the amount of money left (if any) once everyone not in $T$ has had their claims paid in full.

**Theorem 3** *The Talmud solution to the bankruptcy problem is the nucleolus of the game with the above $v(\cdot)$.*



# Conclusions

- There are many 'solution concepts'. E.g., Nash, Kalai-Smorodinski, Shapley, Talmud, consistency, proportional fairness, max-min fairness, etc.
- Any one solution concept will usually violate the axioms associated with some other solution concept.

  If axioms are meant to represent intuition, then counter-intuitive examples are inevitable.
- A 'perfect' solution to a bargaining, arbitration or voting problem is unattainable.

  One must choose a solution concept on the basis of what properties one likes and what counter-intuitive examples one wishes to avoid.