Symmetric Rendezvous Search on K_4

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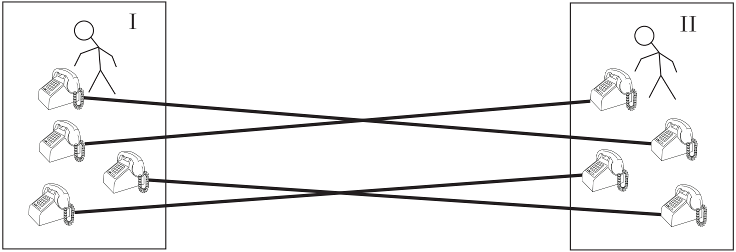
Telephone coordination game (Alpern, 1976)



In each of two rooms, there are n telephones randomly strewn about. The phones are connected pairwise in some unknown fashion. There is a player in each room. In each period $1, 2, \ldots$, each player picks up a phone and says "hello", until the first time that they hear one another. The common aim of the players is to minimize the expected number of periods required to meet.

Symmetric rendezvous search on K_n

Assumptions

1. Two players are randomly placed at two different vertices of the complete graph K_n.
2. There is no commonly held labelling of the vertices.
3. At each of steps, $1, 2, \dots$, each player visits one of the n vertices.
4. The players adopt identical (randomizing) strategies.

What should their common strategy be if they wish to meet in the least expected number of steps?

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Some possible strategies

Move-at-random

If at each discrete step $1, 2, \ldots$ each player were to locate himself at a randomly chosen location, then the expected time to meet would be n. E.g.,

$$ET = 1 + \tfrac{n-1}{n} ET \quad \Longrightarrow \quad ET = n\,.$$

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Wait-for-mommy

Suppose the players could break symmetry (or had some prior agreement). Now it is best for one player to remain stationary while the other tours all other locations in random order. They will meet (on average) half way through the tour. So

$$ET = \tfrac{1}{n-1}\big(1 + 2 + \cdots + (n-1)\big) = \tfrac{1}{2}n\,.$$

The Anderson-Weber strategy

Theorem 1 *In the asymmetric rendezvous search game on K_n the optimal strategy is wait-for-mommy. (Anderson-Weber, 1990)*

Motivated by the optimality of *wait-for-mommy* in the asymmetric case, Anderson and Weber (1990) proposed the following strategy:

AW : If rendezvous has not occurred within the first $(n-1)j$ steps then in the next $n-1$ steps each player should either stay at his initial location or tour the other $n-1$ locations in random order, with probabilities p and $1-p$, respectively, where p is to be chosen optimally.

The Anderson-Weber strategy on K_2

Let $w = \inf\{ET\}$, where the infimum is taken over all possible strategies.

Theorem 2 *On K_2,* **AW** *minimizes $P(T > k)$ for all k.*

Corollary. $w = 2$ on K_2.

AW with $p = \frac{1}{2}$ is the same as *move-at-random*.

The Anderson-Weber strategy on K_3

For K_3 we have (Weber, 2006)

Theorem 3 *On K_3,* **AW** *minimizes ET,*

Moreover it minimizes $E[\min\{T, k\}]$ for all k.

Corollary. $w = \frac{5}{2}$ *on K_3.*

On K_3, **AW** specifies that in each block of two consecutive steps, each player should, independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities $p = \frac{1}{3}$ and $1 - p = \frac{2}{3}$.

AW gives $ET = \frac{5}{2}$, whereas *move-at-random* gives $ET = 3$.

The Anderson-Weber strategy on K_4

On K_4 the expected rendezvous time under **AW** satisfies

$$\begin{aligned} ET &= p^2(3+ET) + 2p(1-p)2 + (1-p)^2\left(\tfrac{1}{2}\tfrac{16}{9} + \tfrac{1}{2}(3+ET)\right) \\ &= \frac{43-14p+25p^2}{9\left(1+2p-3p^2\right)}\,. \end{aligned}$$

The minimum of ET is achieved by taking

$$p = \tfrac{1}{4}\left(3\sqrt{681}-77\right) \approx 0.321983\,,$$

which lead to

$$ET = \tfrac{1}{12}\left(15+\sqrt{681}\right) \approx 3.42466\,.$$

Suppose location 1 (2) is the home location of player I (II).
Each player idependently labels his non-home locations as a, b, c. A tour of non-home locations is one of abc, acb, bac, bca, cab, cba.

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If I has $(a, b, c) = (2, 3, 4)$ and II has $(a, b, c) = (1, 3, 4)$ we find

$$B = \begin{pmatrix} 2 & \mathsf{X} & 3 & \mathsf{X} & \mathsf{X} & 2 \\ \mathsf{X} & 2 & \mathsf{X} & 2 & 3 & \mathsf{X} \\ 3 & \mathsf{X} & 1 & 1 & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & 2 & 1 & 1 & \mathsf{X} & \mathsf{X} \\ \mathsf{X} & 3 & \mathsf{X} & \mathsf{X} & 1 & 1 \\ 2 & \mathsf{X} & \mathsf{X} & \mathsf{X} & 1 & 1 \end{pmatrix}$$

Rows and columns to correspond to abc, acb, bac, bca, cab, cba.
A number shows the step at which players meet.
X indicates that they do not meet.

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Rows and columns to correspond to abc, acb, bac, bca, cab, cba.
A number shows the step at which players meet.
X indicates that they do not meet.
There are 36 such matrices, over which we must average, for each possible pair of assignments by players I and II, of $(2, 3, 4)$ and $(1, 3, 4)$, respectively, to (a, b, c).

$$\begin{pmatrix} 2 & X & 3 & X & X & 2 \\ X & 2 & X & 2 & 3 & X \\ 3 & X & 1 & 1 & X & X \\ X & 2 & 1 & 1 & X & X \\ X & 3 & X & X & 1 & 1 \\ 2 & X & X & X & 1 & 1 \end{pmatrix} \begin{pmatrix} X & 2 & X & 2 & 3 & X \\ 2 & X & 3 & X & X & 2 \\ X & 3 & X & X & 1 & 1 \\ 2 & X & X & X & 1 & 1 \\ 3 & X & 1 & 1 & X & X \\ X & 2 & 1 & 1 & X & X \end{pmatrix} \begin{pmatrix} 3 & X & 2 & X & 2 & X \\ X & 2 & X & 2 & X & 3 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \end{pmatrix} \begin{pmatrix} X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \end{pmatrix} \begin{pmatrix} 2 & X & X & 3 & X & 2 \\ X & 3 & 2 & X & 2 & X \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \end{pmatrix}$$

$$\begin{pmatrix} X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \end{pmatrix} \begin{pmatrix} X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & X & X & X \\ 1 & 1 & X & 2 & X & X \\ 3 & X & 2 & X & 2 & X \\ X & 2 & X & 2 & X & 3 \\ X & X & 2 & X & 1 & 1 \\ X & X & X & 3 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & X & X & 3 & X \\ 1 & 1 & X & X & X & 2 \\ X & 3 & 2 & X & 2 & X \\ 2 & X & X & 3 & X & 2 \\ X & X & 1 & 1 & 2 & X \\ X & X & 1 & 1 & X & 3 \end{pmatrix} \begin{pmatrix} X & X & X & 3 & 1 & 1 \\ X & X & 2 & X & 1 & 1 \\ X & 2 & X & 2 & X & 3 \\ 3 & X & 2 & X & 2 & X \\ 1 & 1 & X & 2 & X & X \\ 1 & 1 & 3 & X & X & X \end{pmatrix} \begin{pmatrix} X & X & 1 & 1 & X & 3 \\ X & X & 1 & 1 & 2 & X \\ 2 & X & X & 3 & X & 2 \\ X & 3 & 2 & X & 2 & X \\ 1 & 1 & X & X & X & 2 \\ 1 & 1 & X & X & 3 & X \end{pmatrix}$$

A new search game on 6 locations

When a player makes a tour in **AW** he chooses it at random. Might something else be better?

Consider a new game, in which at each new step (of 3 old steps) each player makes a tour of his non-home locations.

Let AAB denote three successive tours: the first tour is chosen at random, the second is chosen to be the same as the first, and the third is chosen randomly from amongst the 5 not yet tried.

If successive tours are chosen at random,

$$ET = 1 + \tfrac{1}{2}ET$$

so $ET = 2$.

The optimal 2–Markov policy

Over two steps possible strategies are AA and AB. We find a non-meet matrix of

$$P_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{5} & \frac{13}{50} \end{pmatrix}$$

So

$$ET = p^\top \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{5} & \frac{13}{50} \end{pmatrix} ET \right) p$$

and $\left(\quad \right) \succ 0$. This is minimized by $p^\top = (1/6, 5/6)$, so in fact it is optimal to choose tours at random.

The optimal 3–Markov policy

Now possible strategies over 3 steps are AAA, AAB, ABA, ABB, ABC. The not-meeting matrix is

$$P_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{5} & \frac{13}{50} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} \\ \frac{1}{5} & \frac{2}{25} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} \\ \frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} \\ \frac{1}{20} & \frac{11}{100} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} \end{pmatrix}$$

We find $P_3 \succeq 0$. Again, it turns out that choosing tours at random is optimal, $p^\top = (1, 5, 5, 5, 20)/6^2$.

A 4–Markov policy better than **AW**

Over 4 steps there are 15 possible strategies: $AAAA$, $AAAB$, $AABA$, $AABB$, $AABC$, $ABAA$, $ABAB$, $ABAC$, $ABBA$, $ABBB$, $ABBC$, $ABCA$, $ABCB$, $ABCC$, $ABCD$.

$P_4 =$

$$\begin{pmatrix}
\frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{5} & \frac{1}{5} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & 0 \\
\frac{1}{5} & \frac{13}{50} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{11}{100} & \frac{1}{50} & \frac{1}{50} & \frac{1}{50} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{11}{100} & \frac{11}{100} & \frac{1}{50} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{75} & \frac{2}{25} & \frac{1}{30} & \frac{1}{30} & \frac{1}{30} & \frac{11}{100} & \frac{23}{450} \\
\frac{1}{20} & \frac{11}{100} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{1}{30} & \frac{1}{50} & \frac{7}{150} & \frac{7}{150} & \frac{7}{150} & \frac{1}{20} & \frac{14}{225} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{11}{100} & \frac{1}{50} & \frac{11}{100} & \frac{3}{100} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{2}{75} & \frac{2}{25} & \frac{1}{30} & \frac{1}{30} & \frac{11}{100} & \frac{1}{30} & \frac{23}{450} \\
\frac{1}{20} & \frac{11}{100} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{30} & \frac{1}{50} & \frac{7}{150} & \frac{7}{150} & \frac{1}{20} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{2}{25} & \frac{2}{75} & \frac{1}{30} & \frac{13}{50} & \frac{2}{25} & \frac{11}{100} & \frac{11}{100} & \frac{1}{30} & \frac{1}{30} & \frac{23}{450} \\
\frac{1}{5} & \frac{2}{25} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{2}{25} & \frac{1}{50} & \frac{2}{25} & \frac{13}{50} & \frac{11}{100} & \frac{1}{50} & \frac{11}{100} & \frac{11}{100} & \frac{3}{100} \\
\frac{1}{20} & \frac{11}{100} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{1}{50} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{11}{100} & \frac{7}{50} & \frac{1}{20} & \frac{7}{150} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{11}{100} & \frac{1}{50} & \frac{1}{20} & \frac{7}{50} & \frac{7}{150} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{1}{50} & \frac{11}{100} & \frac{1}{20} & \frac{1}{30} & \frac{11}{100} & \frac{7}{150} & \frac{7}{150} & \frac{7}{50} & \frac{7}{150} & \frac{14}{225} \\
\frac{1}{20} & \frac{1}{50} & \frac{1}{50} & \frac{11}{100} & \frac{1}{20} & \frac{11}{100} & \frac{1}{30} & \frac{7}{150} & \frac{1}{30} & \frac{11}{100} & \frac{7}{150} & \frac{7}{150} & \frac{7}{150} & \frac{7}{50} & \frac{14}{225} \\
0 & \frac{3}{100} & \frac{3}{100} & \frac{23}{450} & \frac{14}{225} & \frac{3}{100} & \frac{23}{450} & \frac{14}{225} & \frac{23}{450} & \frac{3}{100} & \frac{14}{225} & \frac{14}{225} & \frac{14}{225} & \frac{14}{225} & \frac{7}{90}
\end{pmatrix}$$

P_4 has a negative eigenvalue. Choosing tours at random is

$$p^\top = \frac{1}{6^3}(1,5,5,5,20,5,5,20,5,5,20,20,20,20,60)\,.$$

and this gives $ET = 2$. However, using

$$p^\top = \frac{1}{12}(0,1,1,0,0,1,0,0,0,1,0,0,0,0,8)$$

we get $ET = 2 - \frac{23}{16200}$.

Players do $AAAB$, $AABA$, $ABAA$, $ABBB$ each with probability $1/12$, and $ABCD$ with probability $2/3$.

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This is like **AW**. With probability $p = 1/3$ a player does his home tour A and one other tour B. With probability $p = 2/3$ he tours 3 other non-home tours B, C, D.

A strategy better than **AW** for K_4

Consider a 12-Markov strategy consisting of four 3-steps. In each 3-step a player remains home with probability p, or tours his non-home locations with probability $1-p$. It is **AW**, except that when a player makes tours he does so as previously described. Any 1st and 2nd tours sre made at random, but then 3rd and 4th tours are made such that $AAAB$, $AABA$, $ABAA$, $ABBB$ have probabilities $1/12$, and $ABCD$ has probability $2/3$.
There are 1585 possible paths of nonzero probability. Careful computation finds $ET =$

$$\frac{-227773p^8+582884p^7-1329319p^6+1737938p^5-1941235p^4+1420688p^3-998569p^2+389834p-217648}{3\left(82001p^8-218608p^7+327728p^6-315256p^5+215870p^4-104656p^3+36128p^2-8008p-15199\right)}$$

For $p=(1/4)(3\sqrt{681}-77)$ (same as **AW**) this gives ET less than **AW** by 0.00014668.

Conclusion

AW is optimal on K_2 and K_3.

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Anderson and Weber (1990) said that they suspected the strategy might not be optimal, but did not suggest anything better. Fan (2009) showed that **AW** is not optimal in a version of the problem in which players are told that the locations are arranged on a circle, and they are given a common notion of clockwise. He showed that **AW** is optimal for K_n amongst 3–Markov strategies.

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Now we know:

AW is **not** optimal for symmetric rendezvous search on for K_4.