The optimal strategy for symmetric rendezvous search on K_3

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Abstract

In the symmetric rendezvous search game played on K_n (the completely connected graph on n vertices) two players are initially placed at two distinct vertices (called locations). The game is played in discrete steps and at each step each player can either stay where he is or move to a different location. The players share no common labelling of the locations. They wish to minimize the expected number of steps until they first meet. Rendezvous search games of this type were first proposed by Steve Alpern in 1976. They are simple to describe, and have received considerable attention in the popular press as they model problems that are familiar in real life. They are notoriously difficult to analyse. Our solution of the symmetric rendezvous game on K_3 makes this the first interesting game of its type to be solved, and establishes the long-standing conjecture that the Anderson-Weber strategy is optimal.

Keywords: rendezvous search, search games, semidefinite programming

1 Symmetric rendezvous search on K_3

In the symmetric rendezvous search game played on K_n (the completely connected graph on n vertices) two players are initially placed at two distinct vertices (called locations). The game is played in discrete steps, and at each step each player can either stay where he is or move to another location. The players wish to meet as quickly as possible. They use an identical strategy, and this must involve some random moves or else the players will never meet. They have no common labelling of the locations, so a given player must choose the probabilities with which he moves to each of the locations at step k as only a function of where he has been at previous steps.

Let T, w and w_k denote respectively the number of the step on which the players meet, the minimum achievable value of ET, and the minimum achievable value of $E[\min\{T, k+1\}] = \sum_{i=0}^{k} P(T > i)$. We call w the 'rendezvous value' of the game. A long-standing conjecture of Anderson and Weber (1990) is that for symmetric rendezvous search on K_3 the rendezvous value is $w = \frac{5}{2}$. This rendezvous value is achieved by a type of strategy which is now commonly known as the Anderson–Weber strategy (**AW**). For rendezvous search on K_n the **AW** strategy specifies that in blocks of $n-1$ consecutive steps the players should randomize between staying at their initial location and touring the other $n-1$ locations in random

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order. On K_3 this means that in each successive block of two steps, each player should, independently of the other, either stay at his initial location or tour the other two locations in random order, doing these with respective probabilities $\frac{1}{3}$ and $\frac{2}{3}$. The rendezvous value with this strategy is $ET = \frac{5}{2}$.

Rendezvous search problems have a long history. One finds such a problem in the 'Quo Vadis' problem of Mosteller (1965) and recently as 'Aisle Miles' (O'Hare, 2006). The first formal presentation of our problem is due to Alpern (1976), who states it as his 'Telephone Problem'. "Imagine that in each of two rooms, there are n telephones randomly strewn about. They are connected in a pairwise fashion by n wires. At discrete times $t = 0, 1, \ldots,$ players in each room pick up a phone and say 'hello'. They wish to minimize the time t when they first pick up paired phones and can communicate." The **AW** strategy was conjectured to be optimal for K_3 by Anderson and Weber (1990), who proved its optimality for K_2. Subsequently, there have been proofs that **AW** is optimal for K_3 within restricted classes of Markovian strategies, such as those that must repeat in each block of k steps, where k is small, like 2 or 4.

The rest of the paper concerns symmetric rendezvous search on K_3. We have recently shown that the **AW** strategy does not minimize $P(T > k)$ for $k \geq 4$. This is somewhat of a surprise and shows that $ET = \sum_{i=0}^{\infty} P(T > i)$ cannot be minimized by minimizing each term of the sum simultaneously.

However, with Jimmie Fan, we have gained greater computational experience of the problem and have been motivated to make the conjecture that **AW** achieves w_k for all k, i.e., minimizes the truncated sum $\sum_{i=0}^{k} P(T > i)$. In the following section we prove this is true. Our Theorem 1 states that $\{w_k\}_{k=0}^{\infty} = \{1, \frac{5}{3}, 2, \frac{20}{9}, \frac{7}{3}, \frac{65}{27}, \ldots\}$ with $w_k \to \frac{5}{2}$. The symmetric rendezvous game on K_3 becomes the first interesting game of its type to be fully solved.

2 Optimality of the Anderson–Weber strategy

Recall that T denotes the step on which the players meet.

Theorem 1 *The Anderson–Weber strategy is optimal for the symmetric rendezvous search game on K_3, minimizing $E[\min\{T, k+1\}]$ to w_k for all $k = 1, 2, \ldots$, where*

$$w_k = \begin{cases} \frac{5}{2} - \frac{5}{2} 3^{-\frac{k+1}{2}}, & \text{when } k \text{ is odd,} \\ \frac{5}{2} - \frac{3}{2} 3^{-\frac{k}{2}}, & \text{when } k \text{ is even.} \end{cases} \tag{1}$$

Consequently, the minimal achieveable value of ET is $w = \frac{5}{2}$.

Proof Throughout most of the following a subscript k on a vector means that its length is 3^k. A subscript k on a matrix means that it is $3^k \times 3^k$. Let

$$B_k = B_1 \otimes B_{k-1}, \quad \text{where } B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Here '$\otimes$' denotes the Kronecker product. We label the rows and columns of B_1 as $0, 1, 2$. Suppose the locations are arranged on a circle and that players have a common notion of clockwise.[1] Suppose player II is initially placed one position clockwise of player I. Then $B_1(i,j)$ is an indicator for the event that they do not meet when at the first step player I moves i positions clockwise from his initial location, and player II moves j positions clockwise from his initial location. $B^\top$ contains the indicators for the same event, but when player II starts two positions clockwise of player I. Since the starting position of player II is randomly chosen, the problem of minimizing the probability of not having met after the first step is that of minimizing

$$p^\top \big(\tfrac{1}{2}(B_1 + B_1^\top)\big)p\,,$$

over $p \in \Delta$, where $\Delta = \{p : p \geq 0 \text{ and } 1^\top p = 1\}$. Similarly, the 9 rows and 9 columns of B_2 can be labelled as $0,\dots,8$ (base 10), and also $00, 01, 02, 10, 11, 12, 20, 21, 22$ (base 3). The base 3 labelling is helpful, for we may understand $B_2(i_1 i_2, j_1 j_2)$ as an indicator for the event that the players do not meet when at his first and second steps player I moves to locations that are respectively i_1 and i_2 positions clockwise from his initial position, and player II moves to locations that are respectively j_1 and j_2 positions clockwise from his initial position. The problem of minimizing the probability that they have not met after k steps is that of minimizing

$$p^\top \big(\tfrac{1}{2}(B_k + B_k^\top)\big)p\,.$$

In this manner we can also formulate the problem of minimizing $E[\min\{T, k+1\}]$. Let J_k be the $3^k \times 3^k$ matrix that is all 1s and let

$$\begin{aligned} M_1 &= J_1 + B_1 \\ M_k &= J_k + B_1 \otimes M_{k-1} \\ &= J_k + B_1 \otimes J_{k-1} + \cdots + B_{k-1} \otimes J_1 + B_k\,. \end{aligned} \tag{2}$$

Then

$$w_k = \min_{p\in\Delta}\Big\{p^\top M_k p\Big\} = \min_{p\in\Delta}\Big\{\tfrac{1}{2}p^\top (M_k + M_k^\top)p\Big\}\,.$$

It is a difficult problem to find the minimizing p, because $\frac{1}{2}(M_k + M_k^\top)$ is not positive semidefinite once $k \geq 2$. The quadratic form $p^\top\big(\frac{1}{2}(M_k + M_k^\top)\big)p$ has many local minima that are not global minimums. For example, the strategy which randomizes equally over the 3 locations at each step, taking $p^\top = (1, 1 \dots, 1)/3^k$, is a local minimum of this quadratic form.

Consider, for example, $k = 2$. To show that $w_2 = 2$ we must minimize $p^\top(M_2 + M_2^\top)p$. However, the eigenvalues of $\frac{1}{2}(M_2 + M_2^\top)$ are $\{19, \frac{5}{2}, \frac{5}{2}, 1, 1, 1, 1, -\frac{1}{2}, -\frac{1}{2}\}$, so this matrix is not positive semidefinite. In general, the minimization over x of a quadratic form such as $x^\top Ax$ is $\mathcal{NP}$-hard if A is not positive semidefinite. An alternative approach might be to try to show that $\frac{1}{2}(M_2 + M_2^\top) - 2J_2$ is a copositive matrix. For general k , we would wish to show that $x^\top\big(\frac{1}{2}(M_k + M_k^\top) - w_k J_k\big)x \geq 0$ for all $x \geq 0$, where $\{w_k\}_{k=1}^\infty = \{\frac{5}{3}, 2, \frac{20}{9}, \frac{7}{3}, \frac{65}{27}, \dots\}$ are the values obtained by the Anderson–Weber strategy. However, to check copositivity numerically is also $\mathcal{NP}$-hard.

The key idea in this proof is to exhibit a matrix H_k such that $M_k \geq H_k \geq 0$, where H_k is positive semidefinite (denoted $H_k \succeq 0$) and $p^\top H_k p$ is minimized over $p \in \Delta$ to w_k. Since p is

[1] Readers familar with the problem will be aware that it might make a difference whether or not the players are equipped with a common notion of clockwise. We assume that they are. However, since we show that the **AW** strategy cannot be bettered and this strategy makes no use of the clockwise information, **AW** is also optimal if the players do not have a common notion of clockwise.

nonnegative we must have $p^\top M_k p \geq p^\top H_k p \geq w_k$ for all p. For example, we may take

$$M_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 3 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 1 & 1 & 3 & 2 & 3 \end{pmatrix} \geq H_2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 3 & 2 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 1 & 0 & 1 & 3 & 2 & 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 0 & 3 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 3 & 1 & 0 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

where $\frac{1}{2}(H_2 + H_2^\top)$ is positive semidefinite, with eigenvalues $\{18, 3, 3, \frac{3}{2}, \frac{3}{2}, 0, 0, 0, 0\}$. The minimum value of $p^\top H_2 p$ is 2.

We restrict our search for H_k to matrices of a special form. For $i = 0, \ldots, 3^k - 1$ we write $i_{\text{base}\,3} = i_1 \cdots i_k$ (always keeping k digits, including leading 0s when $i \leq 3^{k-1} - 1$); so $i_1, \ldots, i_k \in \{0, 1, 2\}$. We define

$$P_i = P_{i_1 \cdots i_k} = P_1^{i_1} \otimes \cdots \otimes P_1^{i_k}\,,$$

where

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that the subscript is now used for something other than the size of the matrix. It will always be easy for the reader to know the k for which P_i is $3^k \times 3^k$ by context. Observe that $M_k = \sum_i m_k(i) P_i$, where m_k is the first row of M_k. This motivates a search for an appropriate H_k amongst those of the form

$$H_k = \sum_{i=0}^{3^k-1} x_k(i) P_i\,.$$

The condition $M_k \geq H_k$ is equivalent to $m_k \geq x_k$. In the example above, $H_2 = \sum_i x_2(i) P_i$, where $x_2 = (3, 3, 2, 3, 3, 2, 1, 1, 0)$, the first row of H_2.

Let us observe that the matrices $P_0, \ldots, P_{3^k-1}$ commute with one another and so have a common set of eigenvectors. Also, $P_i^\top = P_{i'}$, where $i'_{\text{base}\,3} = i'_1 \cdots i'_k$ is obtained from $i_{\text{base}\,3} = i_1 \cdots i_k$ by letting i'_j be $0, 2, 1$ as i_j is $0, 1, 2$, respectively.

Let the columns of the matrices U_k and W_k contain the common eigenvectors of the $\frac{1}{2}(P_i + P_i^\top)$ and also of $\frac{1}{2}(M_k + M_k^\top)$. The columns of W_k are eigenvectors with eigenvalues of 0. We shall now argue that the condition $(\frac{1}{2}(H_k + H_k^\top) \succeq 0$ is equivalent to $U_k x_k \geq 0$. To see this, note that the eigenvalues of $\frac{1}{2}(H_k + H_k^\top)$ are the same as the real parts of the eigenvalues of H_k. The eigenvectors and eigenvalues of H_k can be computed as follows. Let ω be the cube root of 1 that is $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$. Then

$$V_k = V_1 \otimes V_{k-1}\,, \quad \text{where } V_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

We write $V_k = U_k + iW_k$, and shall make use of the facts that $U_k = U_1 \otimes U_{k-1} - W_1 \otimes W_{k-1}$ and $W_k = U_1 \otimes W_{k-1} + W_1 \otimes U_{k-1}$. It is easily checked that the eigenvectors of P_i are the columns (and rows) of the symmetric matrix V_k and that the first row of V_k is $(1, 1, \ldots, 1)$. The eigenvalues are also supplied in V_k, because if $V_k(j)$ denotes the jth column of V_k (an eigenvector), we have $P_i V_k(j) = V_k(i, j)V_k(j)$. Thus the corresponding eigenvalue is $V_k(i, j)$. Since H_k is a sum of the P_i, we also have $H_k V_k(j) = \sum_i x_i V_k(i, j)V_k(j)$, so the eigenvalue is $\sum_i x_i V_k(i, j)$, or $\sum_i V_k(j, i)x_i$ since V_k is symmetric. Thus the real parts of the eigenvalues of H_k are the elements of the vector $U_k x_k$. This is nonnegative if and only if the symmetric matrix $\frac{1}{2}(H_k + H_k^\top)$ is positive semidefinite.

Let 1_k denote the length 3^k vector of 1s. We will show that we may take $H_k = \sum_i x_k(i)P_i$, where

$$x_1 = (2, 2, 1)^\top$$
$$x_2 = (3, 3, 2, 3, 3, 2, 1, 1, 0)^\top$$

and that we may choose a_k so that for $k \geq 3$,

$$x_k = 1_k + (1, 0, 0)^\top \otimes x_{k-1} + (0, 1, 0)^\top \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)^\top \otimes 1_{k-3}\,. \tag{3}$$

In this construction of x_k the parameter a_k is chosen maximally such that $U_k x_k \geq 0$ and $m_k \geq x_k$.[2] The sum of the components of x_k is

$$1_k^\top x_k = 3^k + 1_{k-1}^\top x_{k-1} + 3^{k-2}(3 + a_k)\,.$$

To prove the theorem we want $1_k^\top x_k/3^k = w_k$, where these are the values specified in (1). This requires the values of the a_k to be:

$$a_k = \begin{cases} 3 - \dfrac{1}{3^{(k-3)/2}}\,, & \text{when } k \text{ is odd,} \\ 3 - \dfrac{2}{3^{(k-2)/2}}\,, & \text{when } k \text{ is even.} \end{cases} \tag{4}$$

So

$$\{a_3, a_4, \ldots, a_{11}, \ldots\} = \{2, \tfrac{7}{3}, \tfrac{8}{3}, \tfrac{25}{9}, \tfrac{26}{9}, \tfrac{79}{27}, \tfrac{80}{27}, \tfrac{241}{81}, \tfrac{242}{81}, \ldots\}\,.$$

Alternatively, the values of $3 - a_k$ are $1, \frac{2}{3}, \frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{2}{27}, \ldots$. For example, with $a_3 = 2$ we have

$$m_3 = (4, 4, 3, 4, 4, 3, 2, 2, 2, 4, 4, 3, 4, 4, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)\,,$$
$$x_3 = (4, 4, 3, 4, 4, 3, 2, 2, 1, 3, 3, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)\,.$$

Note that a_k increases monotonically in k, from 2 towards 3. As $k \to \infty$ we find $a_k \to 3$ and $1_k^\top x_k/3^k \to \frac{5}{2}$. It remains to prove that with these a_k we have $m_k \geq x_k$ and $U_k x_k \geq 0$.

[2] There are many choices of x_2 that work. We can also take $x_2 = (3, 3, 2, 2, 3, 2, 1, 1, 1)$ or $x_2 = (3, 3, 2, 3, 2, 2, 1, 1, 1)$.

$\boldsymbol{m_k \geq x_k}$

To prove $m_k \geq x_k$ is easy; we use induction. The base of the induction is $m_2 = (3,3,2,3,3,2,1,1,1) \geq x_2 = (3,3,2,3,3,2,1,1,0)$. Assuming $m_{k-1} \geq x_{k-1}$, we then have

$$\begin{aligned}
m_k &= 1_k + (1,1,0)^\top \otimes m_{k-1} \\
&\geq 1_k + (1,0,0)^\top \otimes m_{k-1} + (0,1,0)^\top \otimes \left(1_{k-1} + (1,1,0)^\top \otimes 1_{k-2} + (1,1,0,1,1,0,0,0,0)^\top \otimes 1_{k-3}\right) \\
&= 1_k + (1,0,0)^\top \otimes m_{k-1} + (0,1,0)^\top \otimes (3,3,2,3,3,2,1,1,1)^\top \otimes 1_{k-3} \\
&\geq 1_k + (1,0,0)^\top \otimes x_{k-1} \ + (0,1,0)^\top \otimes (a_k,a_k,2,2,a_k,2,1,1,1)^\top \otimes 1_{k-3} \\
&= x_k \,.
\end{aligned}$$

$\boldsymbol{U_k x_k \geq 0}$

To prove $U_k x_k \geq 0$ is much harder. Indeed, $U_k x_k$ is barely nonnegative, in the sense that as $k \to \infty$, $\frac{5}{9}$ of its components are 0, and $\frac{2}{9}$ of them are equal to $\frac{3}{2}$. Thus most of the eigenvalues of $\frac{1}{2}(H_k + H_k^\top)$ are 0. We do not need this fact, but it is interesting that $2\,U_k x_k$ is a vector only of integers.

Let f_k be a vector of length 3^k in which the first component is 1 and all other components are 0. Using the facts that $U_k = U_1 \otimes U_{k-1} - W_1 \otimes W_{k-1} = U_3 \otimes U_{k-3} - W_3 \otimes W_{k-3}$ and $W_k 1_k = 0$ and $U_k 1_k = 3^k f_k$, we have

$$U_2 x_2 = (18, \tfrac{3}{2}, \tfrac{3}{2}, 3, 0, 0, 3, 0, 0)^\top \,,$$

and for $k \geq 3$,

$$\begin{aligned}
U_k x_k &= 3^k f_k + (1,1,1)^\top \otimes U_{k-1} x_{k-1} \\
&\quad + \Big(U_3\left((0,1,0)^\top \otimes (a_k,a_k,2,2,a_k,2,1,1,1)^\top\right)\Big) \otimes U_{k-3} 1_{k-3} \\
&= 3^k f_k + (1,1,1)^\top \otimes U_{k-1} x_{k-1} + 3^{k-3} r_k \otimes f_{k-3} \,, \qquad (5)
\end{aligned}$$

where r_k is

$$\begin{aligned}
r_k &= U_3\left((0,1,0) \otimes (a_k,a_k,2,2,a_k,2,1,1,1)\right)^\top \\
&= \tfrac{3}{2}\big(6+2a_k, 0, 0, a_k-1, 0, a_k-2, a_k-1, a_k-2, 0, \qquad (6) \\
&\qquad -3-a_k, 2-a_k, a_k-2, -a_k, 0, 0, 1, 2-a_k, 0, \qquad (7) \\
&\qquad -3-a_k, a_k-2, 2-a_k, 1, 0, 2-a_k, -a_k, 0, 0\big)^\top . \qquad (8)
\end{aligned}$$

Note that we make a small departure from our subscripting convention, since r_k is not of length 3^k, but of length 27. We use the subscript k to denote that r_k is a function of a_k.

Using (5)–(8) it easy to compute the values $U_k x_k$, for $k = 2, 3, \ldots$. Notice that there is no need to calculate the $3^k \times 3^k$ matrix U_k. Computing $U_k x_k$ as far as $k = 15$, we find that for the values of a_k conjectured in (4) we do indeed always have $U_k x_k \geq 0$. This gives a lower bound on the rendezvous value of $w \geq w_{15} = 16400/6561 \approx 2.49962$. It would not be hard to continue to even larger k (although $U_{15} x_{15}$ is already a vector of length $3^{15} = 14,348,907$). Clearly the method is working. It now remains to prove that $U_k x_k \geq 0$ for all k.

Consider the first third of $U_k x_k$. This is found from (3) and (6) to be

$$3^k f_{k-1} + U_{k-1}x_{k-1} + 3^{k-3}\tfrac{3}{2}\big(6+2a_k, 0, 0, a_k-1, 0, a_k-2, a_k-1, a_k-2, 0\big) \otimes f_{k-3}\,.$$

Assuming $U_{k-1}x_{k-1} \geq 0$ as an inductive hypothesis, and using the fact that $a_k \geq 2$, this vector is nonnegative. So this part of $U_k x_k$ is nonnegative.

As for the rest of $U_k x_k$, notice that r_k is symmetric, in the sense that $S_3 r_k = r_k$, where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $S_3 = S_1 \otimes S_1 \otimes S_1$. The matrix S_k transposes 1s and 2s. Indeed $S_k P_i = P_i^\top$. Thus the proof is complete if we can show that just the middle third of $U_k x_k$ is nonnegative. Assuming that $U_{k-1}x_{k-1} \geq 0$ and $a_k \geq 2$, there are just 4 components of this middle third that depend on a_k and which might be negative. Let I_k denote a $3^k \times 3^k$ identity matrix. This middle third is found from (3) and (7) and is as follows, where we indicate in bold face terms that might be negative,

$$\begin{aligned}&\big((0,1,0)\otimes I_{k-1}\big)U_k x_k \\ &\qquad = U_{k-1}x_{k-1} + \tfrac{3}{2}3^{k-3}\big(\boldsymbol{-3-a_k}, \boldsymbol{2-a_k}, a_k-2, \boldsymbol{-a_k}, 0, 0, 1, \boldsymbol{2-a_k}, 0\big) \otimes f_{k-3}\,.\end{aligned}$$

The four possibly negative components of the middle third are

$$\begin{aligned} t_{k1} &= (0,1,0) \otimes (1,0,0,0,0,0,0,0,0) \otimes f_{k-3}^\top U_k x_k \\ &= (U_{k-1}x_{k-1})_1 + \tfrac{3}{2}3^{k-3}\,(-3-a_k) \end{aligned} \tag{9}$$

$$\begin{aligned} t_{k2} &= (0,1,0) \otimes (0,1,0,0,0,0,0,0,0) \otimes f_{k-3}^\top U_k x_k \\ &= (U_{k-1}x_{k-1})_{3^{k-3}+1} + \tfrac{3}{2}3^{k-3}\,(2-a_k) \end{aligned} \tag{10}$$

$$\begin{aligned} t_{k3} &= (0,1,0) \otimes (0,0,0,1,0,0,0,0,0) \otimes f_{k-3}^\top U_k x_k \\ &= (U_{k-1}x_{k-1})_{3\,3^{k-3}+1} + \tfrac{3}{2}3^{k-3}(-a_k) \end{aligned} \tag{11}$$

$$\begin{aligned} t_{k4} &= (0,1,0) \otimes (0,0,0,0,0,0,0,1,0) \otimes f_{k-3}^\top U_k x_k \\ &= (U_{k-1}x_{k-1})_{7\,3^{k-3}+1} + \tfrac{3}{2}3^{k-3}\,(2-a_k) \end{aligned} \tag{12}$$

The remainder of the proof is devoted to proving that all these are nonnegative. Consider t_{k1}. It is easy to work out a formula for t_{k1}, since

$$\begin{aligned} (U_k x_k)_1 &= f_k^\top U_k x_k \\ &= 3^k + f_{k-1}^\top U_{k-1}x_{k-1} + 3^{k-3}\tfrac{3}{2}(6+2a_k) \\ &= (U_{k-1}x_{k-1})_1 + 4\,3^{k-1} + 3^{k-2}a_k \end{aligned}$$

Thus

$$(U_k x_k)_1 = 2\,3^k + \sum_{i=3}^{k} 3^{i-2}a_i\,, \tag{13}$$

and

$$t_{k1} = \tfrac{1}{2}\,3^k + \sum_{i=3}^{k-1} 3^{i-2}a_i - \tfrac{1}{2}3^{k-2}a_k \tag{14}$$

This is nonnegative since $a_k \le 3$.

Amongst the remaining terms, we observe empirically that $t_{k2} \ge t_{k4} \ge t_{k3}$. It is t_{k3} that is the least of the four terms, and which constrains the size of a_k. Let us begin therefore by finding a formula for t_{k3}. We have

$$\begin{aligned}
t_{k3} &= (0,1,0) \otimes (0,0,0,1,0,0,0,0,0) \otimes f_{k-3}^\top\, U_k x_k \\
&= (0,0,0,1,0,0,0,0,0) \otimes f_{k-3}^\top U_{k-1}x_{k-1} - 3^{k-2}\tfrac{1}{2}a_k \\
&= (0,1,0) \otimes f_1^\top \otimes f_{k-3}^\top\big(3^{k-1}f_{k-1} + (1,1,1)^\top \otimes U_{k-2}x_{k-2} + 3^{k-4}r_{k-1} \otimes f_{k-4}\big) - 3^{k-2}\tfrac{1}{2}a_k \\
&= (1,0,0,0,0,0,0,0,0) \otimes f_{k-4}^\top U_{k-2}x_{k-2} + 3^{k-4}(0,1,0) \otimes f_2)r_{k-1} - 3^{k-2}\tfrac{1}{2}a_k \\
&= (U_{k-2}x_{k-2})_1 - 3^{k-4}\tfrac{3}{2}(3 + a_{k-1}) - 3^{k-2}\tfrac{1}{2}a_k \\
&= (U_{k-2}x_{k-2})_1 - 3^{k-3}\tfrac{1}{2}(3 + a_{k-1}) - 3^{k-2}\tfrac{1}{2}a_k
\end{aligned}$$

This means that t_{k3} can be computed from the first component of $U_{k-2}x_{k-2}$, which we have already found in (13). So

$$\begin{aligned}
t_{k3} &= 2\,3^{k-2} + \sum_{i=3}^{k-2} 3^{i-2}a_i - 3^{k-3}\tfrac{1}{2}(3 + a_{k-1}) - 3^{k-2}\tfrac{1}{2}a_k \\
&= \tfrac{1}{2}3^{k-1} + \sum_{i=3}^{k-2} 3^{i-2}a_i - \tfrac{1}{2}3^{k-3}a_{k-1} - \tfrac{1}{2}3^{k-2}a_k\,. \qquad (15)
\end{aligned}$$

We now put the a_k to the values specified in (4). It is easy to check with (4) and (15) that $t_{k3} = 0$ for all k.

It remains only to check that also $t_{k2} \ge 0$ and $t_{k4} \ge 0$. We have

$$\begin{aligned}
t_{k2} &= (0,1,0) \otimes (0,1,0,0,0,0,0,0,0) \otimes f_{k-3}^\top\, U_k x_k \\
&= (0,1,0,0,0,0,0,0,0) \otimes f_{k-3}^\top U_{k-1}x_{k-1} + 3^{k-2}(1 - \tfrac{1}{2}a_k) \\
&= (1,0,0) \otimes (0,1,0) \otimes f_{k-3}^\top\big(3^{k-1}f_{k-1} + (1,1,1)^\top \otimes U_{k-2}x_{k-2} + 3^{k-4}r_{k-1} \otimes f_{k-4}\big) \\
&\quad + 3^{k-2}(1 - \tfrac{1}{2}a_k) \\
&= (0,1,0) \otimes f_{k-3}^\top U_{k-2}x_{k-2} - 3^{k-4}\tfrac{3}{2}(1 - a_{k-1}) + 3^{k-2}(1 - \tfrac{1}{2}a_k)\,.
\end{aligned}$$

We recognize $(0,1,0) \otimes f_{k-3}^\top U_{k-2}x_{k-2}$ to be the first component of the middle third of $U_{k-2}x_{k-2}$. The recurrence relation for this is

$$\begin{aligned}
(0,1,0) \otimes f_{k-1}^\top U_k x_k &= (0,1,0) \otimes f_{k-1}^\top\big(3^k f_k + (1,1,1)^\top \otimes U_{k-1}x_{k-1} + 3^{k-3}r_k \otimes f_{k-3}\big) \\
&= f_{k-1}^\top U_{k-1}x_{k-1} - 3^{k-2}\tfrac{1}{2}(3 + a_k)\,.
\end{aligned}$$

The right hand side can be computed from (13). So we now have,

$$\begin{aligned}
t_{k2} &= 2\,3^{k-3} + \sum_{i=3}^{k-3} 3^{i-2}a_i - 3^{k-4}\tfrac{1}{2}(3 + a_{k-2}) - 3^{k-3}\tfrac{1}{2}(1 - a_{k-1}) + 3^{k-2}(1 - \tfrac{1}{2}a_k) \\
&= 4\,3^{k-3} + \sum_{i=3}^{k-3} 3^{i-2}a_i - \tfrac{1}{2}3^{k-4}a_{k-2} + \tfrac{1}{2}3^{k-3}a_{k-1} - \tfrac{1}{2}3^{k-2}a_k\,. \qquad (16)
\end{aligned}$$

Finally, we establish a formula for t_{k4}.

$$\begin{aligned}
t_{k4} &= (0,1,0) \otimes (0,0,0,0,0,0,0,1,0) \otimes f_{k-3}^\top U_k x_k \\
&= (0,0,0,0,0,0,0,1,0) \otimes f_{k-3}^\top U_{k-1} x_{k-1} + 3^{k-2}(1 - \tfrac{1}{2} a_k) \\
&= (0,0,1) \otimes (0,1,0) \otimes f_{k-3}^\top \big(3^{k-1} f_{k-1} + (1,1,1)^\top \otimes U_{k-2} x_{k-2} + 3^{k-4} r_{k-1} \otimes f_{k-4}\big) \qquad (17) \\
&\quad + 3^{k-2}(1 - \tfrac{1}{2} a_k) \\
&= (0,1,0) \otimes f_{k-3}^\top U_{k-2} x_{k-2} + 3^{k-4} \tfrac{3}{2} + 3^{k-2}(1 - \tfrac{1}{2} a_k) \\
&= 5\, 3^{k-3} + \sum_{i=3}^{k-3} 3^{i-2} a_i - \tfrac{1}{2} 3^{k-4} a_{k-2} - \tfrac{1}{2} 3^{k-2} a_k \,. \qquad (18)
\end{aligned}$$

Thus we can check the fact that we observed empirically, that $t_{k2} \geq t_{k4} \geq t_{k3}$. We find

$$\begin{aligned}
t_{k2} - t_{k4} &= \tfrac{1}{2} 3^{k-3} (a_{k-1} - 2)\,, \\
t_{k4} - t_{k3} &= \tfrac{1}{2} 3^{k-3} (1 - a_{k-2} + a_{k-1})\,.
\end{aligned}$$

Since a_k is at least 2 and a_k is increasing in k, both of the above are nonnegative. So t_{k2} and t_{k4} are both at least as great as t_{k3}, which we have already shown to be 0. This establishes $U_k x_k \geq 0$ and so the proof is now complete. ∎

3 On discovery of the proof

The proof begs the question: how did we guess the recursion for x_k? Let us restate it here for convenience. With a_k given by (4), the recursion is

$$x_k = 1_k + (1,0,0)^\top \otimes x_{k-1} + (0,1,0) \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1) \otimes 1_{k-3}\,. \qquad (19)$$

Let us briefly describe the steps and ideas in research that led to (19).

We began by computing lower bounds on w_k by solving the semidefinite programming problem

$$\text{maximize } \operatorname{trace}(J_k H_k) \,:\, H_k \leq M_k\,,\; H_k \succeq 0\,. \qquad (20)$$

A similar line of approach has been followed concurrently by Han, Du, Vera and Zuluaga (2006). The lower bounds that are obtained by solving (20) turn out to be achieved by the **AW** strategy and so are useful in proving the Fan–Weber conjecture (that **AW** minimizes $E[\min\{T, k+1\}]$) up to $k = 5$. However, they only produce numerical answers, with little guide as to a general form of solution. In fact, since one can only solve the SDPs up to the numerical accuracy of a SDP solver (which, like *sedumi*, uses interior point methods), such proofs are only approximate. For example, by this method one can only prove that $w_5 \geq 2.40740740$, but not $w_5 = \frac{65}{27} = 2.4\dot{0}\overline{74}$.

We tried to find rational solutions so the proofs could be exact. A major breakthrough was to realise that we could compute a common eigenvector set for $P_1, \ldots, P_{3^k-1}$ and write $M_k = \sum_i m_k(i) P_i$. We discovered this as we noticed and tried to explain the fact that the real parts of all the eigenvalues of $2\,M_k$ are integers. (In fact, there is a little-known theorem which says that if a real symmetric matrix has only integer entries then all its rational eigenvalues must be integers.) This allowed us to recast (20) as the linear program

$$\text{maximize } \sum_{i=0}^{3^k-1} x(i) \,:\, x \leq m_k\,,\; U_k x \geq 0\,. \tag{21}$$

Now we could find exact proofs of the Fan–Weber conjecture as far as $k = 8$, where U_8 is 6561×6561. These solutions were found using *Mathematica*) and were in rational numbers, thus providing us with tight proofs of the optimality of **AW** up to $k = 8$. They also allowed us to prove the Fan-Weber conjecture for greater values of k since the number of decision variables in the LP grows as 3^k, whereas in the SDP it grows as 3^{2k}.

It seems very difficult to find a general solution to (21) that will hold for all k. The LP is highly degenerate with many optimal solutions. There are indeed 12 different extreme point solutions to the LP at just $k = 2$. No general pattern to the solution emerges as it is solved for progressively larger k. For, say $k = 4$, there are many H_4 that can be used to prove $w_k = \frac{7}{3}$. We searched amongst the many solutions for ones with some pattern that might be generalized. This proved very difficult. We tried forcing lots of components of the solution vector to be integers, or identical, and looked for solutions in which the solution vector for $k-1$ was embedded within the solution vector for k. We looked at adding other constraints, and constructed some solutions by augmenting the objective function and choosing amongst possible solution by a minimizing a sum of squares penalty.

Another approach to the problem of minimizing $p^\top M_k p$ over $p \in \Delta$ is to make the identification $Y = pp^\top$. With this identification, Y is positive semidefinite, $\text{trace}(J_k Y) = 1$, and $\text{trace}(M_k Y) = \text{trace}(M_k pp^\top) = p^\top M_k p$. This motivates a semidefinite programming relaxation of our problem: minimize $\text{trace}(M_k Y)$, subject to $\text{trace}(J_k Y) = 1$ and $Y \succeq 0$. This can be recast as the LP

$$\text{minimize } y^\top m_k \,:\, y^\top U_k \geq 0\,,\; 1^\top y = 1\,,\; y \geq 0\,. \tag{22}$$

This is nearly the dual of (21) (which is the same, but has an additional constraint of $y^\top U_k S_k = y^\top U_k$).

With (22) in mind, we imagined taking y as **AW** and worked at trying to guess a full basis in the columns of U_k that is complementary slack to y and from which one can then compute a solution to (21). We also explored a number of different LP formulations. All of this was helpful in building up intuition as to how a general solution might possibly be constructed.

Another major breakthrough was to choose to work with the constraint $x \leq m_k$ in which m_k is the first row of the nonsymmetric matrix M_k, rather than to use the first row of the symmetric matrix $\frac{1}{2}(M_k + M_k^\top)$. By not 'symmetrizing' M_k we were able to find solutions with a simpler form, and felt that there was more hope in being able to write the solution vector x_k in a Kronecker product calculation with the solution vector x_{k-1}. Noticing that all the entries in M_k are integers, we found that it was possible to find a solution for H_k in which all the entries in H_k are integers, as far as $k = 5$. It is not known whether this might be possible for even greater k. The H_k constructed in the proof above have entries that are not integers, although they are of course rational.

Since M_k is computed by Kronecker products it is natural to look for a solution vector of a form in which x_k is expressed in terms of x_{k-1} in some sort formula using Kronecker products. The final breakthrough came in discovering the length 27 vector $(0, 1, 0) \otimes (a_k, a_k, 2, 2, a_k, 2, 1, 1, 1)$. This was found only after despairing of something simpler. We expected that if it were possible to find a Kronecker product form solution similar to (19), then this would use a vector like the above, but of length only 3 or 9. It was only when we hazarded to try something of length 27 that the final pieces fell in place. The final trick was to make the formula for obtaining x_k from x_{k-1} not be constant, but depending on k, as we have done with our a_k. We were lucky at the end that we could solve the recurrence relations for $t_{k1}, t_{k2}, t_{k3}, t_{k4}$ and prove $U_k x_k \geq 0$. It all looks so easy with hindsight!

4 Ongoing research

It is as easy consequence of Theorem 1 that **AW** maximizes $E[\beta^T]$ for all $\beta \in (0,1)$. This follows from the fact that **AW** minimizes $\sum_{i=0}^{k} P(T > i)$ for all k.

We conjecture that **AW** is optimal in a game in which players over-look one another with probability ϵ, (that is, they can fail to meet even when they are in the same location). To analyse this game we simply redefine

$$B_1 = \begin{pmatrix} 1 & 1 & \epsilon \\ \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \end{pmatrix},$$

where $0 < \epsilon < 1$. We can generalize all the ideas in this paper, except that we have not been able to guess a contruction for the matrix H_k.

It will be interesting to explore whether our methods are helpful for other rendezvous problems, set in other graphs, or with more than 3 locations. While for many graphs it is possible to use the solution of a semidefinite programming problem to obtain a lower bound on the rendevzous value, it is not usually possible to recast the SDP as a linear program. A very important feature of the K_3 problem is that it is so strongly captured within the realm of algebra associated with the group of rotational symmetry, whose permuations matrices are the P_i. This continues to be true for rendezvous search on C_n, where locations are arranged in a circle and players have a common notion of clockwise. We are presently looking for results in that direction.

One would still like to have a direct proof that $w = \frac{5}{2}$, without needing to also find the w_k. Perhaps an idea for such a proof is pregnant within the proof above.

Finally, we make some intriguing observations.

1. In the asymmetric version of the rendezvous search game (in which players I and II can adopt different strategies) the rendezvous values for the games on K_2 and K_3 are 1 and 1.5 respectively (and are achieved by the 'wait-for-mommy' strategy). These are exactly 1 less than the rendezvous values of 2 and 2.5 that pertain in the symmetric games (and are achieved by the **AW** strategy).

   A rendezvous search game can also be played on a line. The players start 2 units apart and can move 1 unit left or right at each step. The asymmetric rendezvous value is known to be 3.25 (Alpern and Gal, 1995). In the symmetric game it is known that $4.1820 \leq w \leq 4.2574$ and the conjecture $w = 4.25$ seems plausible (Han, et al. 2006). If that is correct then the difference in rendezvous values for asymmetic and symmetric games is again exactly 1.

2. In the symmetric rendezvous search game played on K_3 it is of no help to the players to be provided with a common notion of clockwise. Similarly, our experience in studying

the symmetric rendezvous search game on the line suggests that it is no help to the players to be provided with a common notion of left and right.

3. No one has yet found a way to prove that the rendezvous value for the symmetric rendezvous search game on K_n is an increasing function of n.

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