Adv. Appl. Prob. **24**, 727–737 (1992)
Printed in N. Ireland

THE INTERCHANGEABILITY OF TANDEM QUEUES WITH HETEROGENEOUS CUSTOMERS AND DEPENDENT SERVICE TIMES

RICHARD R. WEBER,* *University of Cambridge*

Abstract

Consider m queueing stations in tandem, with infinite buffers between stations, all initially empty, and an arbitrary arrival process at the first station. The service time of customer j at station i is geometrically distributed with parameter p_i, but this is conditioned on the fact that the sum of the m service times for customer j is c_j. Service times of distinct customers are independent. We show that for any arrival process to the first station the departure process from the last station is statistically unaltered by interchanging any of the p_i's. This remains true for two stations in tandem even if there is only a buffer of finite size between them. The well-known *interchangeability of* $\cdot/M/1$ *queues* is a special case of this result. Other special cases provide interesting new results.

OUTPUT PROCESSES

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K25
SECONDARY 90B22, 90B15

1. Introduction

Consider m queueing stations in tandem, with infinite buffers between stations, and an arbitrary arrival process at the first station. On completing service at one station a customer joins the queue for service at the downstream station. Suppose that the system is initially empty. It has been known for some time that if customer service times at each station are independent and exponentially distributed, but with different means for each station, then the distribution of the departure process from the final station does not depend on the order of the stations (Weber (1979)). This has been called the *interchangeability of* $\cdot/M/1$ *queues*. This paper generalises that result to a model with dependent service times. For example, an interchangeability result holds when each customer has a service time of 1 at precisely one station—where independently of other customers this is station i with probability p_i—and service times of 0 at all other stations.

Our model shall be in discrete time. Customers have arrival times at the first station and service times at stations $1, \cdots, m$, that are all non-negative integers. Service at each station is first-come-first-served and a customer may not leave station

Received 6 August 1990; revision received 21 October 1991.

*Postal address: University Engineering Department, Management Studies Group, Mill Lane, Cambridge CB2 1RX, UK.

This research has been supported in part by NSF Grant DDM-8914863.

i until all customers who have previously arrived at station i have left it, even if the waiting customer has service time 0 at station i. The service times of distinct customers are independent, but service times at different stations for a given customer may be dependent. The dependency arises in the following way. Let $x_{j,i}$ denote the service time of customer j at station i. Suppose $x_{j,i}$ is a geometrically distributed random variable with parameter $p_i \in (0, 1)$, but the joint distribution of $x_{j,1}, \cdots, x_{j,m}$ is conditioned on the fact that the sum of the $x_{j,i}$'s is c_j. The joint probability distribution is then

$$(1) \qquad P(x_{j,1}, \cdots, x_{j,m}) = \frac{p_1^{x_{j,1}} \cdots p_m^{x_{j,m}}}{\sum\limits_{\substack{k_1, \cdots, k_m \\ k_1 + \cdots + k_m = c_j}} p_1^{k_1} \cdots p_m^{k_m}}, \qquad x_{j,1} + \cdots + x_{j,m} = c_j.$$

It is the p_i's that distinguish the stations from one another. If $p_i > p_k$ then each customer has a greater expected service time at station i than at station k. There is no restriction on the c_j's. These may differ from customer to customer. They may be known numbers, or they may be random variables. It is for this model that we shall prove an interchangeability result: that the departure process from the final station is statistically unaltered by any interchange in the order of the stations. By this we mean that the p_i's may be interchanged without altering the joint distribution of the departure times.

The interchangeability of $\cdot/M/1$ queues has been proved by a number of authors. Our original proof made use of a joint probability generating function for the departure times. Lehtonen's proof (1986) was through matching sample paths for both orders of two servers. Tsoucas and Walrand (1987) employed an interesting random walk identity, and Anantharam (1987) has used a filtering approach.

Other authors have discussed the ordering of stations when service times are not exponential. Friedman (1965) showed that stations with deterministic service times are interchangeable. In general, stations are not interchangeable and the optimal ordering becomes a difficult question. For further discussion of this point see Tembe and Wolff (1974), Pinedo (1982) and Greenberg and Wolff (1988).

It is interesting to consider two or more tandem stations when there are only finite buffers between stations. Avi-Itzhak and Yadin (1965) have found the Laplace transform of the distribution of the time that an arbitrary customer spends in a tandem system when service times are general and arrivals are Poisson. Chao and Pinedo (1990b) have generalized this to batch arrivals. In the case that service times are exponentially distributed Ding and Greenberg (1991) have shown that the order of the last two servers should be arranged so the faster of the two servers comes first. Chao and Pinedo (1990a) have shown that for three stations the output process is the same if the order of the servers is reversed. They have conjectured that this is true for any number of stations in tandem.

In this paper we consider the discrete-time model and actually derive an explicit formula for the joint distribution of the departure times of the first n customers.

When service times are independent and geometrically distributed interchangeability follows from the construction that let to (1). The idea is to condition on the total service time for every customer and then apply the interchangeability result in this paper. This is the same as allowing c_j to have the distribution of the sum of m independent geometric random variables, with parameters $p_1, \cdots, p_m$. In fact, the main point of this paper is to make the observation that for tandem stations with memoryless service distributions interchangeability is a stronger property than previously realised. It holds after conditioning upon the sum of each customer's service times. A recent paper by Kijima and Makimoto (1990) has established a similar result for a two-station model with exponentially distributed service times; it also uses an approach that conditions upon the sum of the service times experienced by each customer.

In Theorem 1 we prove that two stations with a finite intermediate buffer are interchangeable. The proof is as simple as any that has been given in previous papers for the special case of $\cdot/M/1$ queues. Certainly, the number of formulae in this paper is pleasingly small. Theorem 1 generalises the result of Chao et al. (1989) for $\cdot/M/1$ queues. There are other interesting applications, and before proceeding to the proofs in the next section we shall briefly mention two.

The results in this paper might have been formulated and proved in continuous time with no greater difficulty. However, a desire to include one particularly interesting special case has been part of the motivation for preferring discrete time. Suppose that $\sum_{i=1}^{m} p_i = 1$, and $c_j = 1$ for all j. This corresponds to a model in which each customer has a service time 1 at precisely one station, this being station i with probability p_i, and service times of 0 at all other stations. Gideon Weiss has suggested that this might be viewed as a model of queueing in a cafeteria. Each customer moves along the cafeteria counter, pausing to collect those items he desires. Even when a customer has no further items to collect, courtesy prevents him from jumping ahead of anyone who precedes him. In our model, each customer wishes to collect precisely one item and takes time 1 to do so. Or we might mix different types of customer in the same system. Suppose, for example, that customer j has $c_j = 1$ or $c_j = 2$, with probabilities α and $1 - \alpha$ respectively. Here each customer requires either one or two items from the cafeteria.

Another result, to be further discussed in Section 3, is that if the service time $x_{j,i}$ is independently and exponentially distributed with parameter $\mu_j + \lambda_i$, then the time of the nth departure from the mth station has a distribution that is unaltered by the interchange of any of the μ_j's, $1 \leqq j \leqq n$, or λ_i's, $1 \leqq i \leqq m$. This is as if each customer is accompanied by a certain service capacity. Or we might think of two diplomatic legations, of m and n members, that are lined up in two sequences to exchange handshakes. The duration of each handshake depends on the sum of two parameters, one arising from each of the persons involved.

We begin by considering the special case of a two-station system. This really tells the whole story. It will be an easy matter in Section 3 to extend the result to a general number of stations. For two stations an interchangeability result holds even when the buffer space between stations is finite.

2. Interchangeability for two stations

Consider two stations in tandem, with a finite intermediate buffer of size $b \geqq 0$ to hold customers that are waiting to begin service. This means that if a customer completes its service at the first station and b customers are still waiting to begin service at the second station, then the buffer is full and that customer cannot enter the second station, and the first server cannot begin serving another customer, until a customer finishes service at the second station. This is known as *manufacturing blocking*. Suppose customer n arrives at station 1 at time a_n. These arrival times are completely arbitrary, and in general may be random variables. Let $D_{n,i}$ denote the time at which customer n completes service at the ith station and x_n denote its service time at station 1. The service time at station 2 is then $c_n - x_n$. We have

$$D_{n,1} = \max\{a_n, D_{n-1,1}, D_{n-b-2,2}\} + x_n, \tag{2}$$

$$D_{n,2} = \max\{D_{n,1}, D_{n-1,2}\} + c_n - x_n. \tag{3}$$

The effect of blocking due to the finite waiting room is that customer n cannot begin service at station 1 until customer $n - b - 2$ has left station 2. This accounts for the third term within the maximization on the right-hand side of (2). The main theorem of this section is the following

Theorem 1. *Consider two stations in tandem with a finite buffer between stations, and customer service times with the distribution* (1). *The departure process from the second station is statistically identical for both orders of the stations. This holds for any arrival process, even one that depends on the departure process from the last station.*

The truth of Theorem 1 under the assumption that there is only a finite buffer between stations is actually a consequence of the fact that Theorem 1 holds even when the arrival process depends on the departure process. To model the effect of a finite buffer we just consider a modified arrival process, for which the arrival of customer n occurs at time $\max\{a_n, D_{n-b-2,2}\}$. This remark means that it suffices to prove Theorem 1 for the case that the intermediate waiting room is infinite. In a similar manner one may take into account the other generalisations that have been addressed by Chao et al. (1989). For example, we might suppose that any customer who arrives to find more than a certain number of other customers in the system is lost. Theorem 1 is a consequence of the following lemma.

Lemma 1. *Consider the model with an infinite buffer between two stations. Suppose that feasible realizations of the arrival times and the departure times from the* *second station are $a_1, \cdots, a_n$ and $D_{1,2}=d_1, \cdots, D_{n,2}=d_n$ respectively and*

$$d_j > a_j + c_j, \quad \textit{for all } j = 2, \cdots, n. \tag{4}$$

Let $C_n = \sum_{j=1}^n c_j$. Then the following is the only way these departure times can be realised. For every $s \in \{C_n - d_n, \cdots, d_n\}$ there exists a unique corresponding set of service times at station 1, $x(s) = (x_1(s), \cdots, x_n(s))$, such that these departure times are realised, and $\sum_{j=1}^n x_j(s) = s$. Furthermore as s increases from its minimum to its maximum value, $x(s)$ changes in just one component at a time, this being first $x_1(s)$, then $x_2(s)$, and so on. Once s is large enough that the increasing component is of index j or more, the corresponding set of service times is such that station 1 is never idle while it is serving the first j customers.

Note that (4) is the condition that all customers experience some queueing delay. Let us explain how this lemma implies Theorem 1.

Proof of Theorem 1. To prove the theorem it suffices to show that the probability distribution of the departure times of the first n customers is a symmetrical function p_1 and p_2. The proof is by induction on n. Clearly it is true for $n = 1$. Suppose for the moment that the arrival times, $a_1, \cdots, a_n$, are deterministic. Consider a choice of possible values for the departure times, $d_1, \cdots, d_n$. Assume (4) holds and consider a feasible realisation of service times at the first station as $x_1(s), \cdots, x_n(s)$, where $\sum_{j=1}^n x_j(s) = s$. Then the probability of these service times at station 1 is

$$\boldsymbol{P}(d_1, \cdots, d_n; x_1(s), \cdots, x_n(s)) = \prod_{j=1}^n \left\{ p_1^{x_j(s)} p_2^{c_j - x_j(s)} \Big/ \sum_{k=0}^{c_j} p_1^k p_2^{c_j-k} \right\} = \phi(c) p_1^s p_2^{C_n - s} \tag{5}$$

where

$$\phi(c) = 1 \Big/ \prod_{j=1}^n \sum_{k=0}^{c_j} p_1^k p_2^{c_j-k} = \prod_{j=1}^n \left(\frac{p_1 - p_2}{p_1^{c_j+1} - p_2^{c_j+1}} \right)$$

is a normalizing constant and is symmetrical in p_1 and p_2. Since $d_1, \cdots, d_n$ can occur in precisely one way for each $s \in \{C_n - d_n, \cdots, d_n\}$ we need only sum over s in (5) to obtain

$$\boldsymbol{P}(d_1, \cdots, d_n) = \phi(c) \sum_{s=C_n-d_n}^{d_n} p_1^s p_2^{C_n-s}. \tag{6}$$

Because this expression is symmetrical in p_1 and p_2 it follows that the theorem is true.

In the case that (4) does not hold, let j be the smallest index for which it does not hold. If $a_j \geqq d_{j-1}$ then customer j arrives to an empty system and there is an obvious decoupling of customers $j, \cdots, n$ from $1, \cdots, j-1$. It suffices to have proved the theorem for a smaller number of customer departures. Finally, consider the alternative, $a_j < d_{j-1}$. The fact that (4) does not hold implies $d_j = a_j + c_j$ and so customer j waits for service at neither station. This means we must have both of the events $A=[D_{j-1,1}\leqq a_j$ and $d_1,\cdots,d_{j-1}$ are realised] and $B=[d_{j-1}\leqq a_j+x_j]$, where x_j denotes the service time of customer j at the first station. It follows from the lemma that for each $s\in\{C_{j-1}-d_{j-1},\cdots,a_j\}$ there is exactly one choice of $x_1(s),\cdots,x_{j-1}(s)$ with sum s such that A occurs. Thus

$$\boldsymbol{P}(A)=p_2^{d_{j-1}-a_j}\sum_{s=C_{j-1}-d_{j-1}}^{a_j} p_1^s p_2^{C_{j-1}-d_{j-1}+a_j-s}\Big/\prod_{l=1}^{j-1}\sum_{k=0}^{c_l} p_1^k p_2^{c_l-k}.$$

Similarly, given that A occurs, B occurs for $x_j\in\{d_{j-1}-a_j,\cdots,c_j\}$ and thus

$$\boldsymbol{P}(B\mid A)=p_2^{a_j-d_{j-1}}\sum_{x_j=d_{j-1}-a_j}^{c_j} p_1^{x_j}p_2^{c_j+d_{j-1}-a_j-x_j}\Big/\sum_{k=0}^{c_j} p_1^k p_2^{c_j-k}.$$

The probability with which $d_1,\cdots,d_j$ is realized is the product of $\boldsymbol{P}(A)$ and $\boldsymbol{P}(B\mid A)$. From the way these have been written it is easy to see this product is symmetrical in p_1 and p_2. By considering the distribution of x_j conditional on B we see that the departure times of customers $j,\cdots,n$ are distributed as if customer j had arrived to an empty system at time d_{j-1} and had its c_j modified to $c_j-d_{j-1}+a_j$. This also gives a decoupling of customers $j,\cdots,n$ from $1,\cdots,j-1$ and it suffices to have proved the theorem for a smaller number of customer departures.

Consider now a more general arrival process. Note that the above proof holds for any sequence of arrival times. Therefore it holds pathwise when the arrival process is stochastic. Furthermore, so far as the truth of interchangeability goes, the arrival process may be allowed to depend on the departure process. Simply observe that the argument above is equally valid if we consider $a_1,\cdots,a_n$ and $d_1,\cdots,d_n$ to be mutually feasible realizations of the arrival and departure processes, and the arrival process depends upon the departure process, as well as the other way around. This proves Theorem 1.

Remark 1. Of course in the final paragraph of the proof we restrict the arrival process to depend on the departure process in a way that leads to sensible realizations of arrival and departure times. For example, we do not allow $a_j>d_j$, since a customer cannot depart before it arrives. It is not strictly necessary to require it, but the event $[a_j>t]$ should depend only on what can be observed of the departure process up to time t, unless we suppose some clairvoyance. However, one allowable and interesting case, noted in Weber (1979) is a cycle of stations in which a finite number of customers cycle around the stations, the departures from station 2 being the arrivals to station 1.

Remark 2. There is an interesting consequence of Equation (6). Suppose $d_n,a_1,\cdots,a_n$ and $c_1,\cdots,c_n$ are known and are such that (4) holds. (A special case of (4) is when all customers are present at time 0.) The distribution of d_n is obtained by summing over those $d_1,\cdots,d_{n-1}$ that are feasible and hence

$$\boldsymbol{P}(d_1,\cdots,d_{n-1}\mid d_n,a_1,\cdots,a_n,c_1,\cdots,c_n)=1/M,$$

where M is the number of ways that $d_1, \cdots, d_{n-1}$ might be chosen. Thus, knowing the time of the nth departure and the values of $a_1, \cdots, a_n$ and $c_1, \cdots, c_n$, the distribution of previous departure times is uniform over feasible possibilities.

It only remains to prove Lemma 1.

Proof of Lemma 1. The lemma is to be proved for a given set of feasible departure times, $d_1, \cdots, d_n$. Therefore, we may condition on the values of $a_1, \cdots, a_n$. For the reason explained above, it is sufficient to prove the theorem for the case that there is an infinite buffer between stations. The proof is by induction on n. Clearly it is true for $n = 1$. Take as an inductive hypothesis that the lemma is true for $n - 1$. This hypothesis states that $d_1, \cdots, d_{n-1}$ may be realized uniquely by some $(x_1(s), \cdots, x_{n-1}(s))$ for every $s \in \{C_{n-1} - d_{n-1}, \cdots, d_{n-1}\}$. Let $D_{j,1}(s)$ denote the departure time of customer j from the first station when the service times of the first $n - 1$ customers at the first station add up to s, and the departure times of the first $n - 1$ customers from the second station are $d_1, \cdots, d_{n-1}$. Now from (2) we have

$$(7) \qquad D_{n-1,1}(s) = \max_{1 \leqq j \leqq n-1} \left\{a_j + \sum_{k=j}^{n-1} x_k(s)\right\}$$

$$(8) \qquad = \max\left\{a_{n-1} + x_{n-1}, \max_{1 \leqq j \leqq n-2}\left\{D_{j,1}(s) + \sum_{k=j+1}^{n-1} x_k(s)\right\}\right\}.$$

Also, by substituting (2) into (3) and using (4)

$$(9) \qquad d_n = \max\{D_{n-1,1}(s) + c_n, d_{n-1} + c_n - x_n\}.$$

Consider the possibility that the maximum is achieved by the second term within the right-hand side in (9). Since (4) implies $d_n < d_{n-1} + c_n$, the value of x_n for which the maximum is achieved, say $\bar{x} = d_{n-1} - d_n + c_n$, is positive and no more than c_n. By (7) and the inductive hypothesis, $D_{n-1,1}(s)$ is non-decreasing in s. We have assumed that (9) has a solution, so $D_{n-1,1}(s) + c_n$ must be no more than d_n when s takes its minimal value of $C_{n-1} - d_{n-1}$. Imagine increasing s in unit steps from this value until $D_{n-1,1}(s) + c_n$ reaches the value d_n. This must occur before s reaches its maximal value of d_{n-1} since at that point we would have $D_{n-1,1}(d_{n-1}) + c_n \geqq d_{n-1} + c_n > d_n$, by (4). So let $\bar{s}$ be the maximal s for which $D_{n-1,1}(s) + c_n \leqq d_n$ holds. Now suppose we increase s from $\bar{s}$ to $\bar{s} + 1$. This comes about by $x_j(\bar{s} + 1) = x_j(\bar{s}) + 1$, for some $j \leqq n - 1$. By the last part of the inductive hypothesis for the lemma, it follows that for service times $x_1(\bar{s}), \cdots, x_j(\bar{s})$ station 1 is never idle while serving the first j customers. However, from (8), (9) and the fact that $\bar{s}$ is maximal, it follows that station 1 must also not be idle while serving customers $j, \cdots, n$. So $D_{n-1,1}(\bar{s}) = \sum_{j=1}^{n-1} x_j(\bar{s}) = \bar{s}$. Thus $\bar{s} = d_n - c_n$. Therefore, we can satisfy (9) either by taking $x_n = \bar{x}$ and any $s \in \{C_{n-1} - d_{n-1}, \cdots, \bar{s}\}$, or by taking $s = \bar{s}$ and any $x_n \in \{\bar{x}, \cdots, c_n\}$. This means that when $x_n = \bar{x}$ and s takes its minimum value of $C_{n-1} - d_{n-1}$, then $s + x_n$ takes its least possible value, namely $s + x_n = (C_{n-1} - d_{n-1}) + (d_{n-1} - d_n + c_n) =$

$C_n - d_n$, and when $s = \bar{s}$, and x_n takes its maximum value of c_n, then $s + x_n$ takes its greatest possible value, namely $s + x_n = (d_n - c_n) + c_n = d_n$. Using the inductive hypothesis it follows that the possible values of $s + x_n$ are $C_n - d_n, \cdots, d_n$ and each value is achieved by a unique $x_1, \cdots, x_n$. As $s + x_n$ is the notation for $\sum_{j=1}^{n} x_j$, this completes the proof of the lemma.

3. Discussion and conjecture

The generalization to more than two stations is clear. Suppose m stations are in tandem, with infinite buffers between stations and service times distributed as (1). Consider the interchange of station i with the immediately succeeding station, say k. For each customer j, we condition on the value of $x_{j,i} + x_{j,k}$, say $c_{j,k}$. Then $x_{j,i}$ and $x_{j,k}$ are jointly distributed as two geometric random variables that have been conditioned on having sum $c_{i,k}$. The arrival process to the system of two tandem stations, i and k, is the departure process from the upstream stations. By Theorem 1, the departure process from i and k is unaltered when these stations are interchanged. This process is also the arrival process to the system of tandem stations downstream of i and k. In this manner, we see that the departure process from the final station is also unaltered by interchanging i and k. This observation still holds when we remove the conditioning. Therefore by interchanges of adjacent stations, any interchange of stations does not alter the distribution of the departure process. The departure process is statistically identical for all orders of stations. This holds for any arrival process, even one that depends on the departure process from the last station.

A number of authors have shown that in the case that service times at station i are independent and exponentially distributed with parameter λ_i, the departure process is stochastically faster if, given a constraint on $\sum_i \lambda_i$, the λ_i's are made equal. The corresponding result for the model in this paper is that the p_i's should be equal. Unfortunately, we have not been able to make the argument as simple as that for the interchangeability result alone. However, in the special case that all customers are present at the start, and therefore that (4) holds, we can use (6) to show that for any two feasible realizations of the departure times, such that $(d_1', \cdots, d_n') \geqq (d_1, \cdots, d_n)$ componentwise, $P(d_1', \cdots, d_n')/P(d_1, \cdots, d_n)$ is minimized by $p_1 = p_2$. To do this, let $p_1/p_2 = \omega$, and note that

$$\frac{P(d_1', \cdots, d_n')}{P(d_1, \cdots, d_n)} = \frac{\omega^{C_n - d_n'} + \cdots + \omega^{d_n'}}{\omega^{C_n - d_n} + \cdots + \omega^{d_n}}. \tag{10}$$

To show (10) is minimized by $\omega = 1$ it is sufficient to prove this for $d_n' = d_n + 1$, and then think of multiplying together expressions similar to (10). An identity that is useful in verifying that $\omega = 1$ minimizes (10) is that for $i + 2 \leqq j - 2$,

$$\frac{\omega^i + \cdots + \omega^j}{\omega^{i+1} + \cdots + \omega^{j-1}} = \left(\omega + \frac{1}{\omega}\right) - \frac{\omega^{i+2} + \cdots + \omega^{j-2}}{\omega^{i+1} + \cdots + \omega^{j-1}}.$$

The remaining details are straightforward. The fact that

$$P(d_1', \cdots, d_n')/P(d_1, \cdots, d_n)$$

is minimized by $p_1 = p_2$ implies that for any $(d_1, \cdots, d_n)$ the probability that the departure times exceed $(d_1, \cdots, d_n)$ in every component is minimized by $p_1 = p_2$, and thus the distribution of the departure times is stochastically minimized. For the special case mentioned in the introduction, for which $c_i = 1$ for all i, there is a simple proof even with arrivals (see Weber and Weiss (1991)).

A significant new result is obtained by supposing that c_j has the distribution of the sum of m independent geometric distributions, with parameters $\theta_j p_1, \cdots, \theta_j p_m$, $0 < \theta_j < 1$. By multiplying (1) by the probability

$$P(c_j) = \sum_{\substack{k_1, \cdots, k_m \\ k_1 + \cdots + k_m = c_j}} (\theta_j p_1)^{k_1} \cdots (\theta_j p_m)^{k_m} \prod_{i=1}^{m} (1 - \theta_j p_i)$$

we find that the distribution of the service times for customer j are now independent, and geometrically distributed with parameters $\theta_j p_i$. The implication is that if for all j and i, $x_{j,i}$ is independent of all other service times and distributed geometrically with parameter $\theta_j p_i$, then the departure process is unaltered by interchange of any p_i's, provided the order of the customers is not changed.

The continuous-time version of this model is one in which $x_{j,i}$ is distributed independently of other service times as an exponential random variable with parameter $\mu_j + \lambda_i$. (In fact, we move to continuous time by setting $p_i = \exp(-\lambda_i/N)$, $\theta_j = \exp(-\mu_j/N)$, replacing c_j by Nc_j, and letting $N \to \infty$.) The result is that the departure process is unaltered by interchanging the stations provided the order of the customers is not changed. Kijima and Makimoto (1990) have also established this result for the case of two stations in tandem and an infinite intermediate buffer. They also use an argument that conditions on the sum of the two service times that each customer experiences.

Interchanging the notion of a customer and a station leads to a dual view of the tandem queue process in which we think of the stations working their way through the customers. From this viewpoint the times at which the first, second, and successive stations finish serving the last of n customers have a distribution that is independent of the order of those customers. In particular, the time of the nth departure from the mth station is independent of interchanges of any of $\theta_1, \cdots, \theta_n$, and simultaneous interchanges of any of $p_1, \cdots, p_m$.

Consider the continuous-time version of this result, in which $x_{j,i}$ is distributed independently of other service times as an exponential random variable with parameter $\mu_j + \lambda_i$. Assume all customers are present at the start. The time at which the nth customer departs the mth station is

$$\max_{j_1, i_1, \cdots, j_{n+m}, i_{n+m}} x_{j_1, i_1} + \cdots + x_{j_{n+m}, i_{n+m}}, \tag{11}$$

where $(j_1, i_1) = (1, 1)$ and (j_{k+1}, i_{k+1}) is either $(j_k + 1, i_k)$ or $(j_k, i_k + 1)$, subject to $j_k \leqq n$ and $i_k \leqq m$. Think of a rectangular lattice in which a positive unit step may be taken in one coordinate direction at a time. If we say that passing through lattice point (j, i) costs $x_{j,i}$, then (10) is the maximum-cost path from $(1, 1)$ to (n, m). Our result is that the cost of the maximum-cost path has a distribution independent of the interchange of the μ_j's on the rows, or λ_i's on the columns. Of course this suggests the conjecture that there is an interchangeability result in higher dimensions. It is questionable whether this deserves further exploration unless it can be related to some real-life problem. Nonetheless one could consider all paths, in say a three-dimensional lattice, from $(1, 1, 1)$ to (l, m, n), that make only a unit positive step in one coordinate direction at a time, and suppose that the cost of passing through node (i, j, k) is $X_{i,j,k}$, distributed independently of other costs and exponentially with parameter $\alpha_i + \beta_j + \gamma_k$. The conjecture would be that the distribution of the cost of the maximum-cost path is independent of rearrangements within any of the α_i's, the β_j's and the γ_k's.

Acknowledgements

I am grateful to Gideon Weiss for discussion of tandem queueing models in which all service times are 0 and 1 and for commenting on an early proof of interchangeability for that special case. For further details of work on tandem queues with 0–1 service times, including a simple proof of interchangeability for that case, see Weber and Weiss (1991).

References

ANANTHARAM, V. (1987) Probabilistic proof of interchangeability of $/M/1$ queues in series. *QUESTA* **2,** 387–392.

AVI-ITZHAK, B. AND YADIN, M. (1965) A sequence of two servers with no intermediate queue. *Management Sci.* **11,** 553–564.

CHAO, X. AND PINEDO, M. (1990a) On reversibility of tandem queues with blocking. In preparation.

CHAO, X. AND PINEDO, M. (1990b) Batch arrivals to tandem queues without an intermediate buffer. *Stoch. Models* **6,** 735–749.

CHAO, X., PINEDO, M. AND SIGMAN, K. (1989) On the interchangeability and stochastic ordering of exponential queues in tandem with blocking. *Prob. Eng. Inf. Sci.* **3,** 223–236.

DING J. AND GREENBERG, B. S. (1991) Bowl shapes are better with buffers—sometimes. *Prob. Eng. Inf. Sci.* **5,** 159–169.

→ FRIEDMAN, H. D. (1965) Reduction methods for tandem queueing systems. *Operat. Res.* **13,** 121–131.

→ GREENBERG, B. S. AND WOLFF, R. W. (1988) Optimal order of servers for tandem queues in light traffic. *Management Sci.* **34,** 500–508.

KIJIMA, M. AND MAKIMOTO, N. (1990) On interchangeability for exponential single-server queues in tandem. *J. Appl. Prob.* **27,** 690–695.

→ LEHTONEN, T. (1986) On the ordering of tandem queues with exponential servers. *J. Appl. Prob.* **23,** 115–129.

PINEDO, M. (1982) On the optimal order of stations in tandem queues. In *Applied Probability—Computer Science: The Interface*, ed. R. L. Disney and T. J. Ott, pp. 307–326, Birkhauser, Boston, MA.

TEMBE, S. V. AND WOLFF, R. W. (1974) The optimal order of service in tandem queues. *Operat. Res.* **30,** 148–162.

TSOUCAS, P. AND WALRAND, J. (1987) On the interchangeability and stochastic ordering of $\cdot/M/1$ queues in tandem. *Adv. Appl. Prob.* **16,** 515–520.

WEBER, R. R. (1979) The interchangeability of tandem $\cdot/M/1$ queues in series. *J. Appl. Prob.* **16,** 690–695.

WEBER, R. R. AND WEISS, G. (1991) The cafeteria process—tandem queues with dependent 0–1 service times and the bowl shape phenomenon. Submitted.