A PROBLEM OF AMMUNITION RATIONING

At each time, $s=0,1,2,\ldots$ there is a possibility that with probability p a bomber will be overhead. A defender has a stockpile of n units of ammunition. If m, $m \leq n$, of these are fired at a bomber, then it will be destroyed with probability $(1-\alpha^m)$, $0<\alpha<1$.

The objective is to maximize the probability that all bombers arriving within the first t time instants are destroyed. In an obvious notation, the dynamic programming equation is

$$P(n,0) = 1, \text{ and } \quad P(n,t) = (1-p)P(n,t-1) + p \max_{0 \leq m \leq n} (1-\alpha^m)P(n-m,t-1).$$

The following is an intuitively reasonable conjecture.

<u>Conjecture.</u> The optimizing $m(n,t)$ in the above is such that

(A) $m(n,t)$ is nonincreasing in t, and
(B) $m(n,t)$ is nondecreasing in n.

Kinger and Brown (1968) and Samuel(1970) have proved (A). Extensive numerical calculation has found no counterexample to (B). However, (B) remains unproved. There are some interesing variations.

A Continuous Version: Bombers pass overhead in a Poisson process of rate 1. The stockpile of ammunition is x. Firing an amount $y \leq x$ destroys the bomber with probability $1-e^{-y}$. The DP equation is

$$V(x,t) = \max_{0<y\leq x} (1-e^{-y})\{e^{-t} + \int_0^\infty e^{-s} V(x-y,t-s)\, ds\}.$$

We conjecture that $y(x,t)$ is nondecreasing in x.

A Different Objective: Suppose we wish to maximize over an infinite horizon the expected time until a bomber is first missed. The DP equation is $T(n) = 1/p + \max(1-\alpha^m)T(n-m)$. Surprisingly, $m(n)$ is <u>not</u> monotone in n, since for $\alpha=0.5$ we find $m(9)=3$, $m(10)=4$, $m(11)=3$, $m(12)=4$. But we conjecture that $m(x)$ is nondecreasing in x when the ammunition stockpile x is given a continuous model.

A Sufficient Condition for (B): It is easy to check that (B) would follow if $P(n,t)/P(n+1,t)$ could be shown to be nondecreasing in n. We conjecture this is true if t is continuous. But if t is discrete (as in the original statement of the problem) we find $P(14,3)/P(15,3) < P(13,3)/P(14,3)$, for $p=0.54$ and $\alpha=0.65$.

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Richard Weber
(Cambridge)