EVANS, R. V. 1981. Markov Chain Design Problems. *Opns. Res.* **29**, 959–970.
KUMIN, H. J. 1973. On Characterizing the Extrema of a Function of Two Variables, One of Which Is Discrete. *Mgmt. Sci.* **20**, 126–129.
MILLER, B. 1971. On Minimizing Nonseparable Functions Defined on the Integers With an Inventory Application. *SIAM J. Appl. Math.* **21**, 166–185.
ROLFE, A. 1971. A Note on Marginal Allocation in Multiple Server Systems. *Mgmt. Sci.* **17**, 656–658.
WEBER, R. 1983. A Note on Waiting Times in Single Server Queues. *Opns. Res.* **31**, 950–951.
WOLFF, R. W. 1970. Work Conserving Priorities. *J. Appl. Prob.* **7**, 327–337.

A Note on Waiting Times in Single Server Queues

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We show that in any single server queue having a FCFS discipline, the queueing and waiting times of every customer are nonincreasing convex functions of the service rate. An example shows that this need not be the case if the queue has more than one server.

THE PURPOSE of this note is to record a simple proof of the following intuitively reasonable result, which has been formulated by Tu and Kumin [1983].

THEOREM. *In a single server queue having a FCFS discipline the queueing and waiting times of every customer are nonincreasing convex functions of the service rate.*

Proof. Suppose that for a service rate of μ the time required to serve the nth customer is x_n/μ. Let t_n be the time at which the nth customer enters the queue. Let $w_n(\mu)$ denote the time that the nth customer waits until its service is complete and let $q_n(\mu)$ denote the portion of this time that it spends in the queue. The proof of the theorem is by induction on n using the following identities.

$$q_1(\mu) = 0; \qquad w_1(\mu) = x_1/\mu. \tag{1}$$

$$\begin{gathered} q_n(\mu) = \max\{0, (t_{n-1} + w_{n-1}(\mu) - t_n)\}; \\ w_n(\mu) = x_n/\mu + q_n(\mu). \end{gathered} \tag{2}$$

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From (1) we observe that $q_1(\mu)$ and $w_1(\mu)$ are trivially nonincreasing and convex in μ. If two functions are both nonincreasing and convex, then their sum and their maximum also have these properties. So if we assume that $w_{n-1}(\mu)$ is nonincreasing and convex in μ, we can deduce from (2) that $q_n(\mu)$ and $w_n(\mu)$ also have these properties. This completes the inductive proof.

Since the theorem is true for any values of t_n and x_n, it is also true when these are realizations of random variables. In a $G/G/1$ queue the expected time to the nth service completion, and the mean queueing and waiting times (when they exist), are nonincreasing and convex functions of the service rate.

The result is not true when the queue has more than one server. The following is an example of a queue with two servers in which the mean waiting time is not a convex function of μ. Suppose customers arrive at intervals of 4 units and the x_n's are independent identically distributed random variables taking the values 6 and 11 with probabilities 0.67 and 0.33, respectively. Using computer simulation, we have estimated the mean waiting times to be 6.614, 6.531 and 6.445 when μ equals 1.24, 1.25 and 1.26, respectively. As μ increases from 1.24 to 1.25 to 1.26 the mean waiting time decreases by 0.0830 and 0.0856. These estimates each lie in the middle of a 99% confidence interval of length 0.0008. They show that the mean waiting time is not convex in μ. The waiting times can be found by direct calculation if we simplify the example by supposing that $x_{3n} = x_{3n+1} = 6$, $x_{3n+2} = 11$, $(n = 0, 1, 2, \cdots)$. The service times are no longer iid, but we can now calculate that when μ equals 1.24, 1.25 and 1.26 the mean waiting times are 6.463, 6.400 and 6.328, respectively. Again the mean waiting time is not convex in μ.

REFERENCE

TU, H., AND H. KUMIN. 1983. A Convexity Result for a Class of *GI/G/1* Queueing System. *Opns. Res.* **31**, 948–949.