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OPTIMAL SELECTION OF STOCHASTIC INTERVALS UNDER A SUM CONSTRAINT

E. G. COFFMAN, JR., * *AT & T Bell Laboratories*
L. FLATTO, * *AT & T Bell Laboratories*
R. R. WEBER, ** *University of Cambridge*

Abstract

We model a selection process arising in certain storage problems. A sequence $(X_1, \cdots, X_n)$ of non-negative, independent and identically distributed random variables is given. $F(x)$ denotes the common distribution of the X_i's. With $F(x)$ given we seek a decision rule for selecting a maximum number of the X_i's subject to the following constraints: (1) the sum of the elements selected must not exceed a given constant $c > 0$, and (2) the X_i's must be inspected in strict sequence with the decision to accept or reject an element being final at the time it is inspected.

We prove first that there exists such a rule of threshold type, i.e. the ith element inspected is accepted if and only if it is no larger than a threshold which depends only on i and the sum of the elements already accepted. Next, we prove that if $F(x) \sim Ax^\alpha$ as $x \to 0$ for some A, $\alpha > 0$, then for fixed c the expected number, $E_n(c)$, selected by an optimal threshold is characterized by

$$E_n(c) \sim \left[A\left(\frac{\alpha+1}{\alpha}c\right)^\alpha n\right]^{1/1+\alpha} \quad \text{as} \quad n \to \infty.$$

Asymptotics as $c \to \infty$ and $n \to \infty$ with c/n held fixed are derived, and connections with several closely related, well-known problems are brought out and discussed.

DECISION RULE; ON-LINE DECISION; SELECTION POLICY

1. Introduction

Independent, identically distributed random variables are to be selected one at a time from a sequence of length $n \geqq 1$, subject to the constraint that their sum not exceed a given constant $c > 0$. The random variables are non-negative and can be interpreted as interval lengths. For a given distribution, $F(x)$, of interval lengths, the problem is to find a decision rule which maximizes the expected number selected, $E_n(c)$. For the problem of most interest to us, it is understood that in the selection sequence the decision to reject or accept an

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* Postal address: AT & T Bell Laboratories Murray Hill, New Jersey 07974, USA.
** Postal address: Queen's College, Cambridge, CB3 9ET, UK.

interval is made at the time it is inspected, and that such decisions are final. Rules satisfying this constraint are called *on-line*.

Consider an on-line decision rule which selects an interval if and only if it is no larger than the value of a given threshold function $z_m(x)$, where $m \leqq n$ is the number of intervals not yet inspected, and $0 \leqq x \leqq c$ is the difference between the constraint c and the sum of the lengths of the intervals already selected. In our first result we prove that there exists a threshold rule which is optimal among all on-line selection policies. This is done by analyzing a Bellman equation which defines $E_n(c)$ recursively.

We also characterize the asymptotics of $E_n(c)$ as $n \to \infty$. Assuming that $F(x) \sim Ax^\alpha$ as $x \to 0$, where $A,\ \alpha > 0$, we prove that

$$E_n(c) \sim \left[A\left(\frac{\alpha+1}{\alpha}c\right)^\alpha n\right]^{1/1+\alpha} \quad \text{as} \quad n \to \infty.$$

Later, we discuss related problems where $F(x)$ is assumed to be the uniform distribution on $[0, 1]$, i.e. $A = \alpha = 1$. In this case the above function simplifies to $E_n(c) \sim \sqrt{2cn}$.

The proof of the general result is based chiefly on Chernoff estimates [5]. While the precise threshold function of an optimal policy appears to be difficult to find, we shall exhibit a sequence of policies whose threshold functions approach optimality as $n \to \infty$.

It is interesting to compare our results with the naive rule by which an interval is accepted if and only if its length does not increase the sum of selected interval lengths beyond c. While the problem of finding the function $E_n(c)$ explicitly remains difficult in general, results for the special case $c = 1$ and $F(x) = x$ uniform on $[0, 1]$ can be found rather easily. Indeed, the results can be obtained from the analysis of the well-known *record-breaking* problem (a survey can be found in [7]). In a sequential scan of n i.i.d. numbers uniform on $[0, 1]$, how many times is a number encountered which is larger than all previously inspected numbers? It is easy to verify that the two problems are equivalent and that the equivalence breaks down if in our selection problem $F(x)$ is not uniform on $[0, 1]$. An analysis shows that as $n \to \infty$ the number selected is normally distributed with mean and variance $\log n$. The mean $\log n$ is to be compared with our corresponding $\sqrt{2n}$ result.

If we were to allow an initial re-ordering of the i.i.d. sequence, then an optimization rule could simply select the intervals in increasing order of length. This rule will be called the *smallest-first* rule. For given n and c let $N_s \equiv N_{s,c,n}$ denote the number selected by the smallest-first rule. In Section 5 we derive an explicit formula for $E(N_s)$. Clearly, $E(N_s) \geqq E_n(c)$ for all n and c. However, we prove in Section 4 that asymptotically $E(N_s) \sim E_n(c)$ as $n \to \infty$. Thus, the effect of the on-line constraint is asymptotically negligible. In the more picturesque language of Samuels and Steele [11] we have another instance of the fact that 'many stochastic tasks can be performed almost as well by someone unable to see the future (i.e. on-line) as by a prophet. (The smallest-first rule in our case.)

The asymptotics discussed thus far have kept c fixed while $n \to \infty$. We also obtain results for the limit $c \to \infty$, $n \to \infty$ with $c/n = \theta$ held fixed. These limit laws are easier to derive, since they are based only on the law of large numbers.

The problem studied by Samuels and Steele in [11] becomes equivalent to ours in the special case where $F(x)$ is uniform on $[0, 1]$ and $c = 1$. Their problem is to sequentially select from a given i.i.d. sequence a monotone increasing subsequence of maximum expected length. We shall verify this equivalence in the next section by inspection of the Bellman equation for our problem.

As in our case, Samuels and Steele obtain the asymptotic results in two steps: by exhibiting an upper bound and then a sequence of policies which achieves it. In our analysis, the prophet (described by $E(N_s)$) provides the upper bound, but in their problem the prophet does too well, i.e. the prophet's result derived by Logan and Shepp [9] cannot be achieved by any on-line policy. Thus, our simpler upper-bound analysis provides an alternative proof of the Samuels–Steele result.

A problem having a dual relationship with ours is the secretary problem [4], [10] (for other variants see [3], [6]); the difference is essentially in what is adopted as an objective function. The given i.i.d. sequence is now assumed to be infinite and the objective is to minimize the expected number of intervals that have to be inspected in order to find j that sum to at most c. (Here, the interval lengths are the salary demands of secretaries and c is the capital available to pay the j that need to be hired.)

The simple duality of these two problems belies major differences in their analysis. Indeed, whereas the Bellman equation for our primal problem appears to be intractable, the corresponding equation for the dual (secretary) problem can be solved for all n, when $c \leqq 1$ and $F(x) = x^{\alpha}$, $\alpha > 0$. Once again there is a threshold rule that is optimal among the class of on-line policies.

Our asymptotic results are based on less restrictive assumptions about $F(x)$ and c. After formalizing the relationship between the two problems, we show in Section 6 how to use the asymptotic results for the primal problem to obtain new asymptotic results for the dual problem.

Our problem can also be viewed as a special case of a bin-packing problem. In this setting, c is an integer, the intervals are called pieces or items, and the process of selecting is called packing. To our problem one further requirement is added: it must be possible to partition the set of selected intervals into

c blocks, called bins, such that the sum of the lengths of the intervals (packed) in each bin does not exceed 1.

Bruno and Downey [2] have recently analyzed the smallest-first rule extended to this problem under the assumption that interval lengths are uniform random draws from $[0, 1]$. The extended rule simply selects intervals smallest-first for the bins taken one at a time, i.e. intervals for the ith bin are selected smallest-first from the subset of intervals not selected for the first $i-1$ bins.

In contrast to our problem without the partitioning constraint, the smallest-first algorithm does not necessarily maximize the number of intervals selected. However, in the uniform case Bruno and Downey prove that the algorithm is asymptotically optimal in probability as $n \to \infty$. They also give detailed results on the rate of convergence. It is easy to verify that, without the partitioning constraint, at most c more intervals can be selected smallest-first from a given sequence. Thus, for fixed c the partitioning constraint has a negligible effect as $n \to \infty$, and our asymptotic result specialized to the uniform case (i.e. $E(N_s) \sim \sqrt{2cn}$) also applies to the bin-packing version.

Extension of our asymptotic results for on-line rules is also easily worked out for the bin-packing problem. In particular, it is easy to modify the threshold selection rule so as to satisfy the partitioning constraint and sacrifice at most c items in so doing.

A principal application of our model is to problems of storage. Such problems exist within a large variety of industrial settings, whenever objects must be packed efficiently in one dimension. As a concrete example, in a computer system it may be required to allocate to main memory a maximum subset of some collection of records or files.

2. The Bellman equation and an optimal threshold rule

Let $E_n(x)$ denote the maximal expected number of selected intervals which sum to no more than x, where $x \geqq 0$ and $n \geqq 0$ are real and integer-valued, respectively. We develop a Bellman equation for $E_n(x)$ as follows. If the first of $n+1$ intervals has a length exceeding x, which happens with probability $1-F(x)$, then it must be rejected, and the maximal (conditional) expected number of selected intervals must be $E_n(x)$. But if the first of the $n+1$ intervals has length t, $0 \leqq t \leqq x$, then the maximal (conditional) expected number of selected intervals will be $E_n(x)$ or $1 + E_n(x-t)$ according as the interval is rejected or accepted, respectively. Hence, for $x, n \geqq 0$

$$E_{n+1}(x) = (1 - F(x))E_n(x) + \int_0^x \max\,(E_n(x),\, 1 + E_n(x-t))\,dF(t). \tag{1}$$

We note here that if $E_n(x-t)$ is replaced by $E_n(t)$ then we have the Bellman equation for the *monotone subsequence* problem of Samuels and Steele (see (2.5) in [11]). This replacement is valid if $F(x)$ is uniform on $[0,1]$, so the equivalence of the two problems is immediate in this case.

Now assume that $F(0)=0$, and $F(x)$ is continuous for $F(x)\geqq 0$ and strictly increasing wherever $F(x)<1$.

Theorem 1. Let $n\geqq 1$ and $x\geqq 0$. Then $E_n(x)$ satisfies the following:

(i) $E_n(0)=0$.

(ii) $E_n(x)$ is continuous for $x\geqq 0$.

(iii) If $F(x)<1$ for $x\geqq 0$, then $E_n(x)$ is strictly increasing for $x\geqq 0$. If $F(x)<1$ for $x\leqq s$ and $F(x)=1$ for $x\geqq s$, then $E_n(x)$ is strictly increasing for $0\leqq x\leqq ns$ and $E_n(x)=n$ for $x\geqq ns$.

The proof of Theorem 1 follows in a straightforward manner from the Bellman equation (1), and we omit it.

The threshold function $z_{n+1}=z_{n+1}(x)$, $x\geqq 0$, is defined by

$$z_{n+1}(x)=\begin{cases} x, & \text{if } E_n(x)\leqq 1,\\ \text{unique solution to} & \\ E_n(x)-E_n(x-t)=1, \quad 0\leqq t\leqq x, & \text{if } E_n(x)\geqq 1.\end{cases} \tag{2}$$

Observe that the uniqueness of t in (2) is guaranteed by Theorem 1. We have $E_n(x)<1+E_n(x-t)$ if $0\leqq t<z_{n+1}(x)$, and $1+E_n(x-t)<E_n(x)$, if $z_{n+1}(x)<t\leqq x$. Thus we have a threshold rule under the optimal policy: the first of the $(n+1)$ intervals is accepted if its length is not greater than $z_{n+1}(x)$, and rejected if its length exceeds $z_{n+1}(x)$. We may rewrite (1) as

$$\begin{aligned} E_{n+1}(x) &= (1-F(x))E_n(x)+\int_0^{z_{n+1}}(1+E_n(x-t))\,dF(t)\\ &\quad+\int_{z_{n+1}}^{x}E_n(x)\,dF(t)\\ &=(1-F(z_{n+1}))E_n(x)+F(z_{n+1})+\int_0^{z_{n+1}}E_n(x-t)\,dF(t).\end{aligned} \tag{3}$$

E_n and z_n are obtained recursively from (2) and (3). Unfortunately, the computations become difficult even for small n. We list below the first few values of E_n and z_n in the uniform case:

$$F(x)=\begin{cases} x, & 0\leqq x\leqq 1\\ 1, & x\geqq 1,\end{cases}$$

$$E_0(x)=0,\qquad E_1(x)=\begin{cases} x, & 0\leqq x\leqq 1\\ 1, & x\geqq 1\end{cases},\qquad E_2(x)=\begin{cases} 2x-x^2/2, & 0\leqq x\leqq 2\\ 2, & x\geqq 2,\end{cases}$$

$$z_1(x)=z_2(x)=x.$$

For the uniform case it is easy to prove that $z_n(x)$ is increasing in x and $E_n(x)$ is concave in x. In general, however, $z_n(x)$ need not be monotone in x.

Heuristically, the asymptotic formula for $E_n(c)$ may be derived as follows. Think of the threshold $z_n(c)$ as being practically constant in n and denote it by $\varepsilon > 0$. Thus, we accept only those of the n intervals whose length is at most ε. The number of such intervals is $N \sim nF(\varepsilon)$ and the average length of an accepted interval is $\mu(\varepsilon)/F(\varepsilon)$, where $\mu(\varepsilon) = \int_0^\varepsilon x \, dF(x)$. Thus, the sum of the lengths of accepted intervals is $S \sim n\mu(\varepsilon)$. Since they are to sum to no more than c, we must have $n\mu(\varepsilon) \leqq c$, and so

$$\varepsilon \leqq \mu^{-1}\left(\frac{c}{n}\right), \qquad E(N) \sim nF(\varepsilon)$$

Thus, it becomes plausible to choose $\varepsilon = \mu^{-1}(c/n)$ and to conjecture that

$$E_n(c) \sim n(F \, . \, \mu^{-1})\left(\frac{c}{n}\right). \tag{4}$$

We shall in fact verify the above asymptotic formula for a large class of distribution functions. However, we have no proof that the formula holds for all distribution functions, nor do we have a counterexample.

Apart from the general results in Theorem 1, it seems difficult to obtain further properties of $E_n(c)$, e.g. asymptotics as $n \to \infty$, directly from the Bellman equation in (1). We have found it necessary to follow another approach based on the Chernoff estimates provided in the next section. The approach is indirect in the following sense. We analyze the smallest-first rule, which is obviously optimal in the class of *all* rules, and we find in Section 4 the asymptotics of $E(N_s)$. We then define a threshold rule and show that the expected number it selects is asymptotically the same as $E(N_s)$. Thus, the given rule must be asymptotically optimal in the class of on-line policies, and its asymptotic behavior the same as that of $E_n(c)$. This approach can also be applied to the secretary problem, as shown in Section 6.

3. Chernoff estimates

Let $X_1, \cdots, X_n$ be the lengths of the n i.i.d. intervals with $F(x) = P(X_i \leqq x)$, and recall the notation $\mu(\varepsilon) = \int_0^\varepsilon x \, dF(x)$. For $\varepsilon > 0$ the following random variables will be useful in acquiring information about N_s and $E_n(c)$:

N_ε = the number of X_i's not exceeding ε

S_ε = the sum of the X_i's counted by N_ε

$\tilde{S}_\varepsilon$ = the sum of the X_i's, given that $N_\varepsilon = n$.

We shall show in Sections 4 and 6 that for a large class of distribution functions, it is very likely that the above random variables are close to their expected values when $\varepsilon = \mu^{-1}(c/n)$ and n is large. These results will follow from the probability estimates of Theorem 3 later in this section.

We have

$$N_\varepsilon = \sum_{i=1}^{n} \chi_i, \qquad S_\varepsilon = \sum_{i=1}^{n} Z_i, \qquad \tilde{S}_\varepsilon = \sum_{i=1}^{n} W_i \tag{5}$$

where the random variables in each sum are i.i.d. and

$$P(\chi_i = 1) = F(\varepsilon), \qquad P(\chi_i = 0) = 1 - F(\varepsilon)$$

$$P(Z_i \leqq z) = \begin{cases} 1 - F(\varepsilon) + F(z), & 0 \leqq z \leqq \varepsilon \\ 1, & z \geqq \varepsilon \end{cases}$$

$$P(W_i \leqq w) = \begin{cases} F(w)/F(\varepsilon), & 0 \leqq w \leqq \varepsilon \\ 1, & w \geqq \varepsilon. \end{cases}$$

The expectations of χ_i, Z_i, W_i are given by

$$E(\chi_i) = F(\varepsilon), \qquad E(Z_i) = \mu(\varepsilon) = \int_0^\varepsilon x\, dF(x), \qquad E(W_i) = \nu(\varepsilon) = \frac{\mu(\varepsilon)}{F(\varepsilon)}.$$

Theorem 2. Let $Y_1, \cdots, Y_n, \cdots$ be i.i.d. with $E(Y_i) = 0$ and suppose $|E(Y_i^n)| \leqq Mr^n$, $2 \leqq n < \infty$, for some $M, r > 0$. Then

$$P\left[\left|\sum_{i=1}^{n} Y_i\right| > m\right] \leqq 2 \exp\left(-\frac{m^2}{4Mr^2 n}\right), \quad \text{if} \quad 0 \leqq m \leqq 2Mrn. \tag{6}$$

Proof. Let $Y = \sum_{i=1}^{n} Y_i$ and let λ and m be two arbitrary non-negative numbers. Since $\exp(\lambda(Y - m)) \geqq 1$ when $Y - m \geqq 0$, we have the Chernoff bound

$$P[Y > m] \leqq E[\exp(\lambda(Y - m))] = (\phi(\lambda))^n \exp(-\lambda m) \equiv \psi(\lambda), \tag{7}$$

where

$$\phi(\lambda) = E(\exp(\lambda Y_i)) = 1 + \theta(\lambda), \qquad \theta(\lambda) \equiv \sum_{n=2}^{\infty} \frac{\lambda^n E(Y_i^n)}{n!}. \tag{8}$$

We have

$$|\theta(\lambda)| \leqq M \sum_{n=2}^{\infty} \frac{(\lambda r)^n}{n!} \leqq M(\lambda r)^2, \quad \text{if} \quad \lambda r \leqq 1, \tag{9}$$

so that

$$\psi(\lambda) \leqq (1 + Mr^2\lambda^2)^n \exp(-\lambda m) \leqq \exp(Mr^2 n\lambda^2 - \lambda m), \quad \text{if} \quad \lambda r \leqq 1. \tag{10}$$

Letting $\lambda = m/2Mr^2 n$, we conclude from (7) and (10) that

$$P[Y > m] \leqq \exp\left(-\frac{m^2}{4Mr^2 n}\right), \quad \text{if} \quad 0 \leqq m \leqq 2Mrn. \tag{11}$$

Replacing Y_i by $-Y_i$, we conclude from (11) that

$$(12)\qquad P[Y<-m]\leqq \exp\left(-\frac{m^2}{4Mr^2n}\right),\quad \text{if}\quad 0\leqq m\leqq 2Mrn.$$

The inequality in (6) now follows from (11) and (12).

We remark in passing that there is a sizeable literature on bounds of the type given in Theorem 2. The bounds we have chosen are reasonably simple and adequate for our needs. For refinements we refer the interested reader to [1], [8].

For $n\geqq 2$, we have the following central moment estimates

$$(13)\qquad \begin{aligned} &E(|\chi_i-E\chi_i|^n)\leqq E(|\chi_i-E\chi_i|^2)=F(\varepsilon)(1-F(\varepsilon))\leqq F(\varepsilon)\\ &E(|Z_i-EZ_i|^n)=(1-F(\varepsilon))(\mu(\varepsilon))^n+\int_0^\varepsilon |z-\mu(\varepsilon)|^n\,dF(z)\\ &\qquad\qquad\leqq (1-F(\varepsilon))(\varepsilon F(\varepsilon))^n+\varepsilon^n F(\varepsilon)\leqq 2F(\varepsilon)\varepsilon^n\\ &E(|W_i-EW_i|^n)\leqq \varepsilon^n. \end{aligned}$$

In Theorem 2 let Y_i successively be $\chi_i-E\chi_i$, Z_i-EZ_i and W_i-EW_i. From (13) we find that the hypotheses of Theorem 2 hold when we replace (M, r) by $(F(\varepsilon), 1)$, $(2F(\varepsilon), \varepsilon)$, and $(1, \varepsilon)$, respectively. Thus, the following result is proved.

Theorem 3. We have the bounds

$$(14)\qquad P[|N_\varepsilon-F(\varepsilon)n|>m]\leqq 2\exp\left(-\frac{m^2}{4F(\varepsilon)n}\right)\quad \text{if}\quad 0\leqq m\leqq 2F(\varepsilon)n$$

$$(15)\qquad P[|S_\varepsilon-\mu(\varepsilon)n|>m]\leqq 2\exp\left(-\frac{m^2}{8F(\varepsilon)\varepsilon^2n}\right)\quad \text{if}\quad 0\leqq m\leqq 4F(\varepsilon)\varepsilon n$$

$$(16)\qquad P[|\tilde{S}_\varepsilon-\nu(\varepsilon)n|>m]\leqq 2\exp\left(-\frac{m^2}{4\varepsilon^2n}\right)\quad \text{if}\quad 0\leqq m\leqq 2\varepsilon n.$$

4. Limit laws for N_s, $E(N_s)$ and $E_n(c)$

Let $Y_1^n\leqq Y_2^n\leqq\cdots\leqq Y_n^n$ be the order statistics of $X_1,\cdots,X_n$. For given $c>0$, let N_s be the largest $j\leqq n$ such that $Y_1^n+\cdots+Y_j^n\leqq c$; i.e. N_s is the number of intervals summing to no more than c as selected by the smallest-first policy. We shall now derive limit laws for N_s and $E(N_s)$.

We assume that $F(x)$ is continuous and strictly increasing wherever $F(x)<1$, and that $F(x)\sim Ax^\alpha$ as $x\to 0$, where $A, \alpha>0$. Under these assumptions we have the following result.

Lemma 1.

$$\mu(x) \sim \frac{A\alpha}{\alpha+1} x^{\alpha+1}, \qquad \mu^{-1}(x) \sim \left(\frac{1}{A}\frac{\alpha+1}{\alpha} x\right)^{1/\alpha+1} \quad \text{as} \quad x \to 0,$$

$$(17) \qquad (F \,.\, \mu^{-1})(x) \sim f(x) = \left(A^{1/\alpha}\frac{\alpha+1}{\alpha} x\right)^{\alpha/\alpha+1} \quad \text{as} \quad x \to 0,$$

$$\nu(x) \sim \frac{\alpha}{\alpha+1} x, \quad \nu^{-1}(x) \sim \frac{\alpha+1}{\alpha} x \quad \text{as} \quad x \to 0,$$

$$(F \,.\, \nu^{-1})(x) \sim (f(x))^{1+\alpha} \quad \text{as} \quad x \to 0.$$

Proof. Integration by parts gives $\mu(x) = xF(x) - \int_0^x F(t)\, dt$. The asymptotics follow from this formula by routine calculus. It is easy to verify that the assumption that $F(x)$ is continuous and strictly increasing implies the same property for $\mu(x)$ and $\nu(x)$; thus $\mu^{-1}(x)$ and $\nu^{-1}(x)$ are well defined for small $x \geqq 0$.

Let $\varepsilon = \mu^{-1}(c/n)$, $p = F(\varepsilon)$. These are well defined for n large, say $n \geqq n_0$, so let us assume from now on that $n \geqq n_0$. By Lemma 1

$$(18) \qquad \varepsilon \sim \frac{(f(c)/A)^{1/\alpha}}{n^{1/1+\alpha}}, \qquad p \sim \frac{f(c)}{n^{\alpha/1+\alpha}} \quad \text{as} \quad n \to \infty.$$

Let $N'_{c,n} = N_\varepsilon$ and $S'_{c,n} = S_\varepsilon$. In (14) and (15) choose m to be $(pn)^{\frac{3}{4}}$ and $\varepsilon(pn)^{\frac{3}{4}}$, respectively. From (18), $pn \to \infty$ as $n \to \infty$, so that the conditions $m \leqq 2pn$ and $m \leqq 4\varepsilon pn$ of (14) and (15) are fulfilled for large n. We have $n\mu(\varepsilon) = c$, and conclude from Theorem 3 that the following bounds apply.

Theorem 4. For given $c > 0$ and n sufficiently large,

$$(19) \qquad P[|N'_{c,n} - pn| > (pn)^{\frac{3}{4}}] \leqq 2\exp\left(-\tfrac{1}{4}(pn)^{\frac{1}{2}}\right)$$

$$(20) \qquad P[|S'_{c,n} - c| > \varepsilon(pn)^{\frac{3}{4}}] \leqq 2\exp\left(-\tfrac{1}{8}(pn)^{\frac{1}{2}}\right).$$

With the help of Theorem 4 we can now prove the following.

Theorem 5.

(i) For given $c > 0$, $\delta > 0$ and $\beta = 1/3(1+\alpha)$ we have

$$(21) \qquad P\left[\left|\frac{N_s}{f(c)n^{1/\alpha+1}} - 1\right| > \delta\right] = O(\exp(-n^\beta)),$$

where the multiplicative constant hidden in the $O(\cdot)$ notation depends on δ and c but not n.

(ii) For given $c > 0$,

$$(22) \qquad E(N_s) \sim f(c)n^{1/\alpha+1} \quad \text{as} \quad n \to \infty.$$

Proof. The idea of the proof is as follows. From (18)–(20) we conclude that it is very likely that $N'_{c,n} \sim f(c)n^{1/\alpha+1}$ and $S'_{c,n} \sim c$ as $n \to \infty$. Since $S'_{c,n} = Y_1^n + \cdots + Y_{N'_{c,n}}^n$, it follows that it is very likely that $N_s \sim f(c)n^{1/\alpha+1}$ as $n \to \infty$. The argument below formalizes this idea.

Proof of (i). Choose $0 < c_1 < c < c_2$ so that

$$\left|\frac{f(c_i)}{f(c)} - 1\right| < \delta, \qquad i = 1, 2. \tag{23}$$

Let

$$\varepsilon_i = \mu^{-1}\left(\frac{c_i}{n}\right), \qquad p_i = F(\varepsilon_i) \tag{24}$$

and for $i = 1, 2$ define the events

$$D_i \equiv D_{c_i,n} = [|N'_{c_i,n} - p_i n| \leqq (p_i n)^{\frac{3}{4}}] \cap [|S'_{c_i,n} - c_i| \leqq \varepsilon_i (p_i n)^{\frac{3}{4}}]. \tag{25}$$

From (18) we have

$$\varepsilon_i (p_i n)^{\frac{3}{4}} \sim 0, \qquad (p_i n) \sim f(c_i) n^{1/\alpha+1} \quad \text{as} \quad n \to \infty, \quad i = 1, 2. \tag{26}$$

Hence, if D_1 occurs and n is large, then $S'_{c_1,n} < c$ and

$$N_s \geqq N'_{c_1,n} \geqq p_1 n - (p_1 n)^{\frac{3}{4}} > (1 - \delta) f(c) n^{1/\alpha+1}. \tag{27}$$

Similarly, if D_2 occurs and n is large, then $S'_{c_2,n} > c$ and

$$N_s \leqq N'_{c_2,n} \leqq p_2 n + (p_2 n)^{\frac{3}{4}} < (1 + \delta) f(c) n^{1/\alpha+1}. \tag{28}$$

For any event D, let $\bar{D}$ denote its complement. From (19) and (20)

$$P(\overline{D_1 \cap D_2}) = O(\exp(-n^\beta)). \tag{29}$$

The result in (21) now follows from (27)–(29).

Proof of (ii). Define the event

$$D = [|(N_s/f(c)n^{1/\alpha+1}) - 1| < \delta].$$

By (27) and (28) $D_1 \cap D_2 \subset D$ for n large. Using $N_s \leqq n$, we can write

$$(1 - \delta) f(c)^{1/1+\alpha}[1 - p(\bar{D})] \leqq E(N_s) \leqq (1 + \delta) f(c) n^{1/\alpha+1} + nP(\bar{D}). \tag{30}$$

Together, (29) and (30) imply

$$1 - \delta \leqq \liminf_{n\to\infty} \frac{E(N_s)}{f(c)n^{1/1+\alpha}} \leqq \limsup_{n\to\infty} \frac{E(N_s)}{f(c)n^{1/1+\alpha}} \leqq 1 + \delta. \tag{31}$$

Letting $\delta \to 0$, we obtain (22).

Theorem 6.

$$\lim_{n\to\infty} \frac{E_n(c)}{f(c)n^{1/\alpha+1}} = 1.$$

Proof. Let $0 < c_1 < c$ and define ε_1 and p_1 as in the proof of Theorem 5. Let T be the threshold policy that accepts only intervals of size at most ε_1 and selects these whenever the sum of the intervals selected does not exceed c_1. Let D_1 be defined by (25). By Theorem 4

$$P(D_1) \geqq 1 - 4\exp\left(-\tfrac{1}{8}(p_1 n)^{\frac{3}{4}}\right), \quad \text{if } n \text{ is large.} \tag{32}$$

If D_1 occurs and n is large, then $S'_{c_1,n} < c_1$ so the $N'_{c_1,n}$ intervals sum to no more than c_1. Hence,

$$N_T = N'_{c_1,n} \geqq p_1 n - (p_1 n)^{\frac{3}{4}} \quad \text{if } D_1 \text{ occurs and } n \text{ is large.} \tag{33}$$

Together (32) and (33) yield

$$E(N_T) \geqq \left(p_1 n - (p_1 n)^{\frac{3}{4}}\right)\left(1 - 4\exp\left(-\tfrac{1}{8}(p_1 n)^{\frac{1}{2}}\right)\right), \quad \text{if } n \text{ is large,} \tag{34}$$

and therefore

$$\liminf_{n\to\infty} \frac{E(N_T)}{f(c)n^{1/\alpha+1}} \geqq \frac{f(c_1)}{f(c)}. \tag{35}$$

Since $E(N_T) \leqq E_n(c) \leqq E(N_s)$, we conclude from (22) and (35) that

$$\frac{f(c_1)}{f(c)} \leqq \liminf_{n\to\infty} \frac{E_n(c)}{f(c)n^{1/\alpha+1}} \leqq \limsup_{n\to\infty} \frac{E_n(c)}{f(c)n^{1/\alpha+1}} \leqq 1.$$

The limit $c_1 \to c$ produces the desired result.

5. A formula for $E(N_s)$ in the uniform case

We begin with the following general lemma.

Lemma 2. Let $p_{nk} = P(Y_1^n + \cdots + Y_k^n \leqq c)$. Then

$$E(N_s) = \sum_{k=1}^{n} p_{nk}. \tag{36}$$

Proof. Define the indicator function $\chi_k = 1$ if $Y_1^n + \cdots + Y_k^n \leqq c$, and $\chi_k = 0$ otherwise. Then

$$E(N_s) = E\left(\sum_{k=1}^{n} \chi_k\right) = \sum_{k=1}^{n} E(\chi_k) = \sum_{k=1}^{n} p_{nk}. \tag{37}$$

We shall now obtain an explicit formula for the p_{nk}, and hence $E(N_s)$, in the uniform case: $F(x) = x$, $0 \leqq x \leqq 1$, and $F(x) = 1$, $x \geqq 1$.

Theorem 7. Let $F(x)$ be uniform on $[0, 1]$ and assume $c \leqq 1$. Then

$$E(N_s) = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^{k-i}(i-c)^{k-1}}{(i-1)!\,(k-i)!}\left[1 - \left(\frac{i-c}{c}\right)^{n-k+1}\right]. \tag{38}$$

Remark. The proof determines the p_{nk} by using a well-known relation between the uniform distribution and the Poisson process. For this reason, the proof cannot be generalized to distributions other than the uniform. For $c > 1$ it is also possible to obtain a formula for p_{nk}, but it is more complicated. Since the methods are similar, we leave the calculations to the interested reader. We remark that we have not been able to deduce the asymptotics of $E(N_s)$ from formula (38).

Proof. Let T_k denote the time of the kth event in a Poisson process with parameter 1. Let Z_k denote the duration of the kth interevent interval. Then $T_k = Z_1 + \cdots + Z_k$, where $P[Z_k \geqq t] = \exp(-t)$, $t \geqq 0$. Since $F(x)$ is uniform on $[0, 1]$, we know that the order statistics $Y_1^n, \cdots, Y_n^n$ and the random variables $T_1/T_{n+1}, \cdots, T_n/T_{n+1}$ have the same joint distribution. Hence,

$$(39)\qquad \begin{aligned} p_{nk} &= P(T_1 + \cdots + T_k \leqq cT_{n+1}) \\ &= P[kZ_1 + (k-1)Z_2 + \cdots + Z_k \leqq c(Z_1 + \cdots + Z_{n+1})]. \end{aligned}$$

Renaming the i.i.d. random variables $Z_1, \cdots, Z_k$ as $Z_k, \cdots, Z_1$ we can write

$$(40)\qquad \begin{aligned} p_{nk} &= P[Z_1 + 2Z_2 + \cdots + kZ_k \leqq c(Z_1 + \cdots + Z_{n+1})] \\ &= P\left[\left(\frac{1-c}{c}\right)Z_1 + \cdots + \left(\frac{k-c}{c}\right)Z_k \leqq Z_{k+1} + \cdots + Z_{n+1}\right]. \end{aligned}$$

Now denote $Z_1, \cdots, Z_k$ by $U_1, \cdots, U_k$ and $Z_{k+1}, \cdots, Z_{n+1}$ by $V_1, \cdots, V_{n-k+1}$, respectively. Let

$$U = \sum_{i=1}^{k} \frac{i-c}{c} U_i, \qquad V = \sum_{i=1}^{n-k+1} V_i.$$

The U_i's and V_i's are i.i.d. with $P(U_i > t) = P(V_i > t) = e^{-t}$, $k \geqq 0$, and by (40)

$$(41)\qquad p_{nk} = P(U \leqq V).$$

The density function for V is well known and is given by

$$(42)\qquad g(v) = \frac{v^{n-k}e^{-v}}{(n-k)!}, \qquad v \geqq 0.$$

Let $f(u)$ be the density function for U. Then

$$(43)\qquad \begin{aligned} \int_0^\infty \exp(\lambda u) f(u)\, du = E(\exp(\lambda U)) &= \prod_{i=1}^{k} E\left(\exp\left(\lambda \frac{i-c}{c} U_1\right)\right) \\ &= \prod_{i=1}^{k} \frac{1}{1-\left(\frac{i-c}{c}\right)\lambda}, \qquad \lambda < \frac{c}{k-c}. \end{aligned}$$

Expanding into partial fractions and taking inverse Laplace transforms we conclude that

$$f(u) = \sum_{i=1}^{k} \frac{cR_i}{i-c} \exp\left(-\frac{c}{i-c}u\right), \qquad u \geqq 0. \tag{44}$$

where

$$R_i = (i-c)^{k-1} \prod_{\substack{1 \leqq j \leqq k \\ j \neq i}} \frac{1}{i-j} = \frac{(-1)^{i-1}(i-c)^{k-1}}{(i-1)!\,(k-1)!}.$$

From (41), (42) and (44) we obtain

$$\begin{aligned} p_{nk} &= \int_0^\infty \int_0^v f(u)g(v)\,du\,dv \\ &= \sum_{i=1}^{k} \frac{(-1)^{k-i}(i-c)^{k-1}}{(i-1)!\,(k-i)!}\left[1-\left(\frac{i-c}{i}\right)^{n-k+1}\right] \end{aligned} \tag{45}$$

whereupon (38) follows by summing over k.

6. The dual (secretary) problem

Recall that in the dual problem the i.i.d. intervals of an infinite sequence are to be inspected one at a time until j are found which sum to no more than c. The objective is to minimize the expected number of inspections, $\hat{E}_j(c)$, needed. In [4] a threshold rule is proved optimal among on-line selection policies.

Let $X_1, \cdots, X_n, \cdots$ denote the interval lengths and, for a fixed $1 \leqq n < \infty$, let $Y_1^n \leqq \cdots \leqq Y_n^n$ denote the order statistics of $X_1, \cdots, X_n$. Define

$$M_j = \text{smallest } n \geqq j \text{ such that } Y_1^n + \cdots + Y_j^n \leqq c, \quad 1 \leqq j < \infty$$

$$N_n = \text{largest } j,\ 1 \leqq j \leqq n, \text{ such that } Y_1^n + \cdots + Y_j^n \leqq c, \quad 1 \leqq n < \infty.$$

Observe that N_n is the random variable called N_s in Section 4; it is the number of intervals out of the first n which sum to no more than c under the smallest-first rule. M_j is the number of intervals which need to be inspected so that at least j of them sum to at most c. Clearly, $E(M_j) \leqq \hat{E}_j(c)$, just as $E(N_s) \geqq E_n(c)$ in the primal problem. Simple algorithms achieving $E(M_j)$ are easily designed, although they will not satisfy the on-line constraint.

Even though $E(M_j) \leqq \hat{E}_j(c)$, we shall show, in analogy with the primal problem, that $\hat{E}_j(c) \sim E(M_j)$ as $j \to \infty$. Thus, the sequential threshold rule for the dual problem is asymptotically optimal over the class of all selection algorithms.

We begin with the asymptotics of M_j, which are based on the observation that

$$P(M_j > n) = P(N_n < j), \qquad P(M_j \leqq n) = P(N_n \geqq j), \qquad 1 \leqq j \leqq n < \infty. \tag{46}$$

Theorem 8. Let $\delta > 0$. Then

$$P\left(\left|\frac{M_j}{\left(\frac{j}{f(c)}\right)^{\alpha+1}} - 1\right| > \delta\right) = O(\exp(-j^{\frac{1}{4}}), \tag{47}$$

where $f(c)$ is defined as in (17).

Proof. It suffices to consider $0 < \delta < 1$. Let

$$n = \left\lfloor (1+\delta)\left(\frac{j}{f(c)}\right)^{\alpha+1} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer no larger than x. Then

$$j < f(c)\left(\frac{1}{1+\delta/2}\right)^{1/\alpha+1} n^{1/\alpha+1} \leqq n, \quad j \text{ large}. \tag{48}$$

By (21), (46) and (48) we have for large j

$$\begin{aligned} &P\left(M_j > (1+\delta)\left(\frac{j}{f(c)}\right)^{\alpha+1}\right) = P(M_j > n) = P(N_n < j) \\ &\leqq P\left(N_n < \left(\frac{1}{1+\delta/2}\right)^{1/\alpha+1} f(c) n^{1/\alpha+1}\right) = O(\exp(-n^{\beta})) = O(\exp(-j^{\frac{1}{4}}). \end{aligned} \tag{49}$$

Similarly, let

$$n = \left\lfloor (1-\delta)\left(\frac{j}{f(c)}\right)^{\alpha+1} \right\rfloor.$$

Then

$$f(c)\left(\frac{n}{1-\delta}\right)^{1/\alpha+1} \leqq j \leqq f(c)\left(\frac{2n}{1-\delta}\right)^{1/\alpha+1} \leqq n, \quad j \text{ large}. \tag{50}$$

By (21), (46) and (50) we have for large j

$$\begin{aligned} &P\left(M_j \leqq (1-\delta)\left(\frac{j}{f(c)}\right)^{\alpha+1}\right) = P(M_j \leqq n) = P(N_n \geqq j) \\ &\leqq P\left(N_n \geqq \left(\frac{1}{1-\delta}\right)^{1/\alpha+1} f(c) n^{1/\alpha+1}\right) = O(\exp(-n^{\beta})) = O(\exp(-j^{\frac{1}{4}})), \end{aligned}$$

and (47) is therefore proved.

Theorem 9. We have

$$E(M_n) \sim \hat{E}_n(c) \sim \left(\frac{n}{f(c)}\right)^{1+\alpha} \quad \text{as} \quad n \to \infty. \tag{51}$$

Remark. The asymptotic formula for $E(M_n)$ is intuitive as it follows from formula (22) by replacing n and $E(N_s)$ respectively by $E(M_n)$ and n. Similarly, the asymptotic formula for $\hat{E}_n(c)$ follows from the one for $E_n(c)$.

Proof. Since $E(M_n) \leqq \hat{E}_n(c)$, it suffices to show that

$$\liminf_{n\to\infty} \frac{E(M_n)}{\left(\frac{n}{f(c)}\right)^{1+\alpha}} \geqq 1 \tag{52}$$

and

$$\limsup_{n\to\infty} \frac{\hat{E}_n(c)}{\left(\frac{n}{f(c)}\right)^{1+\alpha}} \leqq 1. \tag{53}$$

To prove (52) define the event

$$D = D_{\delta,n} = \left[\left| \frac{M_n}{\left(\frac{n}{f(c)}\right)^{1+\alpha}} - 1 \right| \leqq \delta \right], \qquad 0 < \delta < 1.$$

By Theorem 8, we have for fixed δ

$$P(\bar{D}) = O(\exp(-n^{\frac{1}{4}})), \tag{54}$$

and therefore

$$E(M_n) \geqq [1 - P(\bar{D})](1-\delta)\left(\frac{n}{f(c)}\right)^{1+\alpha}. \tag{55}$$

From (54) and (55) we get

$$\liminf_{n\to\infty} \frac{E(M_n)}{\left(\frac{n}{f(c)}\right)^{1+\alpha}} \geqq 1 - \delta \tag{56}$$

so that (52) is obtained by letting $\delta \to 1$.

To prove (53) let $0 < c_1 < c_2 < c_3 < c$ and $\varepsilon = v^{-1}(c_1/n)$. Let T be the following two-stage threshold policy. From the n intervals T begins by accepting only those whose lengths do not exceed ε, so long as the accepted interval lengths do not sum to greater than c_2. The second stage of T begins when an interval of length no greater than ε would increase the sum beyond c_2. From that point on T only accepts intervals with lengths not exceeding $(c - c_3)/n$. Note that at the end of the first stage, the sum of the accepted interval lengths is at most $c_2 + \varepsilon < c_3$ (for n large). Hence, n intervals summing to no more than c can be selected under policy T.

Let M be the number of interval inspections required by T to select n intervals. To estimate M and $E(M)$, we introduce the mutually independent sequences of i.i.d. random variables $\{\tau_i\}$, $\{\tau_i'\}$ and $\{W_i\}$, where τ_i and τ_i' have geometric distributions with parameters $F(\varepsilon)$ and $F((c-c_3)/n)$, respectively, and

$$P(W_i \leqq x) = \frac{F(x)}{F(\varepsilon)}, \qquad 0 \leqq x \leqq \varepsilon, \qquad P(W_i \leqq x) = 1, \quad x > \varepsilon.$$

Define the event $D = [W_1 + \cdots + W_n \leqq c_2]$ and let $\chi_{\bar{D}}$ be the indicator function of $\bar{D}$. Then

$$(57) \qquad \begin{aligned} M &= \sum_{i=1}^{n} \tau_i \quad \text{if } D \text{ occurs} \\ M &\leqq \sum_{i=1}^{n} \tau_i + \sum_{i=1}^{n} \tau_i' \quad \text{if } \bar{D} \text{ occurs.} \end{aligned}$$

Hence,

$$M \leqq \sum_{i=1}^{n} \tau_i + \left(\sum_{i=1}^{n} \tau_i + \sum_{i=1}^{n} \tau_i'\right)\chi_{\bar{D}}.$$

The independence assumption implies

$$(58) \qquad \begin{aligned} E(M) &\leqq E\left(\sum_{i=1}^{n} \tau_i\right) + E\left(\left[\sum_{i=1}^{n} \tau_i + \sum_{i=1}^{n} \tau_i'\right]\chi_{\bar{D}}\right) \\ &= \frac{n}{F(\varepsilon)} + \left[\frac{n}{F(\varepsilon)} + \frac{n}{F\left(\frac{c-c_3}{n}\right)}\right]P(\bar{D}). \end{aligned}$$

We must now estimate $P(\bar{D})$. Since $n\nu(\varepsilon) = c_1$, we obtain from Theorem 3 (see (16))

$$(59) \qquad P(|\tilde{S}_\varepsilon - c_1| \geqq m) \leqq 2\exp\left(-\frac{m^2}{4n\varepsilon^2}\right), \quad \text{if} \quad 0 \leqq m \leqq 2n\varepsilon.$$

Choose $m = \varepsilon n^{\frac{3}{4}}$. From (17) $\lim_{n\to\infty} \varepsilon n^{\frac{3}{4}} = 0$, and thus we conclude from (59) that

$$(60) \qquad P(\bar{D}) \leqq P(\tilde{S}_\varepsilon \geqq m + c_1) \leqq 2\exp\left(-\tfrac{1}{4}n^{\frac{1}{2}}\right), \quad n \text{ large.}$$

Also, from (17)

$$(61) \qquad \frac{n}{F(\varepsilon)} \sim \left(\frac{n}{f(c_1)}\right)^{1+\alpha}, \qquad \frac{n}{F\left(\frac{c-c_3}{n}\right)} \sim \frac{n^{1+\alpha}}{A(c-c_3)^{\alpha}}.$$

Together, (58), (60), and (61) imply

$$\limsup_{n\to\infty} \frac{E(M)}{\left(\frac{n}{f(c_1)}\right)^{1+\alpha}} \leqq 1,$$

and since $\hat{E}_n(c) \leqq E(M)$ we obtain

$$\limsup_{n\to\infty} \frac{\hat{E}_n(c)}{\left(\frac{n}{f(c_1)}\right)^{1+\alpha}} \leqq 1.$$

Letting $c_1 \to c$ we get the desired result in (53).

7. Asymptotics for $c = n\theta$

Returning to our original problem it is of interest to consider asymptotic results when $n \to \infty$ and $c \to \infty$ with $c/n = \theta$ fixed. While the methods of earlier sections are adequate to handle the new limit, they are in fact unnecessary; the only asymptotic result that we need is the law of large numbers.

For $c = \theta n$ define the threshold $\varepsilon = \mu^{-1}(\theta)$. As before, let N_ε be the number of intervals no larger than ε and let S_ε be their sum. We now assume the existence of the first moment

$$m = \int_0^\infty x\, dF(x),$$

where $F(x)$ is any continuous distribution with $F(0) = 0$ which is strictly increasing wherever $F(x) < 1$. Note that, among the distributions having first moments, we are dealing with a broader class of distributions than previously.

Recalling that $\alpha(\theta) = F \cdot \mu^{-1}(\theta)$, we have by the law of large numbers

$$\frac{N_\varepsilon}{n} \xrightarrow{\text{D}} \alpha(\theta), \qquad \frac{S_\varepsilon}{n} \xrightarrow{\text{D}} \theta \quad \text{as} \quad n \to \infty. \tag{62}$$

The proof of Theorem 5, with obvious modifications, shows that (62) implies the following result.

Theorem 10. Let $0 < \theta < m$ and $c = \theta n$. Then

$$\frac{N_s}{\alpha(\theta) n} \xrightarrow{\text{D}} 1 \quad \text{as} \quad n \to \infty, \tag{63}$$

$$E(N_s) \sim \alpha(\theta) n \quad \text{as} \quad n \to \infty, \tag{64}$$

$$E_n(\theta n) \sim \alpha(\theta) n \quad \text{as} \quad n \to \infty. \tag{65}$$

We remark that the theorem also holds for $\theta \geqq m$ provided we take $\alpha(\theta) = 1$ for $\theta \geqq m$. This follows from $N_s \leqq n$, $E_n(\theta n) \leqq n$ and a standard continuity argument applied to the function $\alpha(\theta)$.

It seems to us that a similar result can be proved for M_n, $E(M_n)$ and $\hat{E}_n(\theta n)$ introduced in Section 6. However, the modifications of the argument of Section 6 are not very clear and we have not worked out the details. For another approach to this problem see [4].

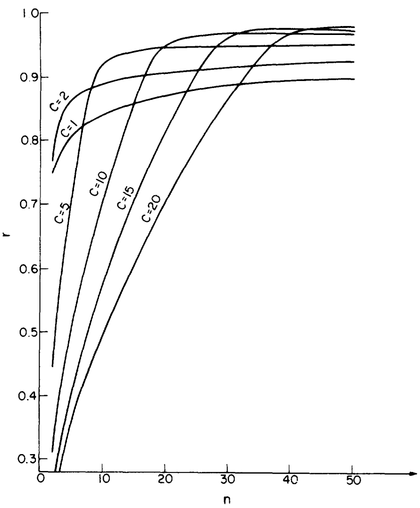

Figure 1. Plot of $r = E_n(c)/\sqrt{2cn}$ for various values of c

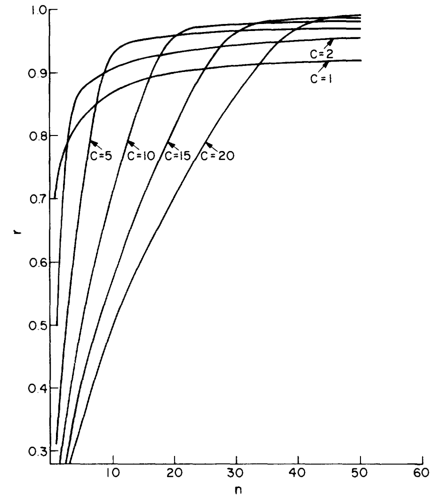

Figure 2. Plot of $r = E(N_{s,c,n})/\sqrt{2cn}$ for various values of c

8. Final remarks

We conclude with numerical results designed to assess the approximation inherent in asymptotics and the sacrifice made in the on-line constraint. Figures 1 and 2 plot the 'performance ratios' $r = E_n(c)/\sqrt{2cn}$ and $E(N_s)/\sqrt{2cn}$ for several values of c under the assumption that interval lengths are uniformly distributed over $[0, 1]$. The results for $E_n(c)$ were obtained by iterative solution of the Bellman equations in (1). Because of numerical problems encountered in evaluating expressions like (38) when n is large, the data for $E(N_s)$ were obtained by routine Monte Carlo simulations.

An obvious feature of each of the plots in both figures is the relatively sharp 'knee', which occurs at approximately $n = 2c$. If $n \leqq c$ then all intervals will be selected and $E_n(c) = n$. If $c < n \leqq 2c$, then the expected number of intervals selected will not be much less than n, since $n = 2c$ items of average length sum to no more than c. Thus, in the region $[0, 2c]$ the asymptotic value $\sqrt{2cn}$ will have little meaning.

Beyond the knees the curves become rather flat, and show that r increases with c as expected. For example, for all $c \geqq 2$ and $n \geqq 15$ the curves are within 10% of the asymptotic value 1, but for fixed c very large increases in n are required to approach 1 substantially closer than the values just beyond the knee.

Figure 3 compares the smallest-first rule with the optimal threshold policy by pairing off three of the plots in Figures 1 and 2. Depending on the performance

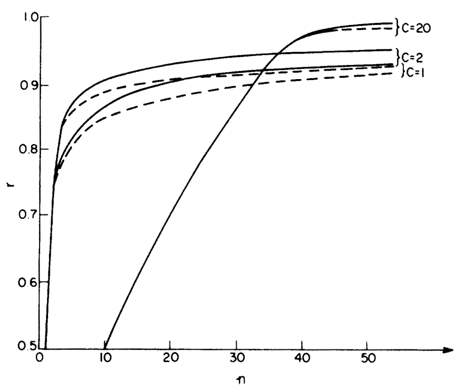

Figure 3. Plots of $E_n(c)/\sqrt{2cn}$ (dashed lines) and $E(N_s)/\sqrt{2cn}$ (solid lines)

standards of the application, the sacrifice incurred by the on-line constraint may be quite tolerable. Note, however, that although the curves for both the smallest-first and the optimal threshold policy have the same asymptotic value for the same value of c, the difference between them approaches 0 very slowly.

A problem left open by our asymptotic analysis is the specification of an optimal threshold function, $z_n(x)$, for fixed n and $x \leqq c$. As a numerical example, we have shown in Figure 4 a plot of the optimal threshold function for $c = 1$, and items uniformly distributed over $[0, 1]$. The plot is based on a recursive evaluation of the Bellman equation in (1). Although the figure shows a monotonic approach from below of $z_n(1)/\sqrt{2/n}$ to 1, we have no proof of this property. Note that the approach of $z_n(1)/\sqrt{2/n}$ to 1 appears to be rather faster than the approach of $E_n(1)/\sqrt{2n}$ to 1 in Figure 3.

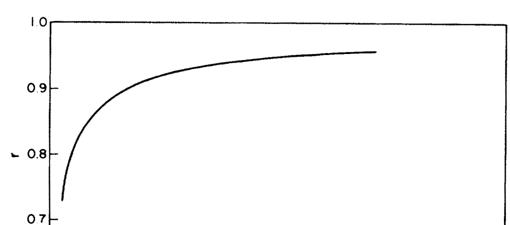

Figure 4. Graph of $r = z_n(1)/\sqrt{2/n}$

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