D2: Optimization
Examples Sheet

[*This example sheet with comments for supervisors is circulated by the Statistical Laboratory to Directors of Studies and those supervisors on our mailing list (contact 37959 to be added to this list). Any comments or corrections gratefully received. The student example sheet with which this sheet corresponds can be identified by the year and version number at the top of this page (it is produced from the same source file and so the question parts are identical). Supervisors may make copies of this sheet available to students if they think it would be helpful.*]

Comments and corrections to `c.sparrow@statslab.cam.ac.uk` please. Questions for Lectures 1–6 and 7–12 may be suitable for 1st and 2nd supervisions. Questions F1–F18 are suitable for further practice and revision. A copy of last years sheet (identical to this one) can be found on the WWW site: `http://www.statslab.cam.ac.uk/~rrw1/opt/index.html`.

Linear programming: introduction (Lectures 1–2)

Questions in this section are introductory and cover, for a different problem, the same material as the first two lectures of the course. If you attended lectures (and understood them) you may wish to omit some or all of these questions.

1. Let P be the linear problem

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & - & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0. \end{array}$$

Solve P graphically in the $x_1,\ x_2$ plane.

[*Optimum 8/3 at $x_1 = x_2 = 4/3$ (vertex c). The feasible set has 5 vertices.*]

2. Write P in matrix notation: max $c^T x$ subject to $Ax \leq b,\ x \geq 0$ and write down the problem D defined by: min $b^T \lambda$ subject to $A^T \lambda \geq c,\ \lambda \geq 0$.

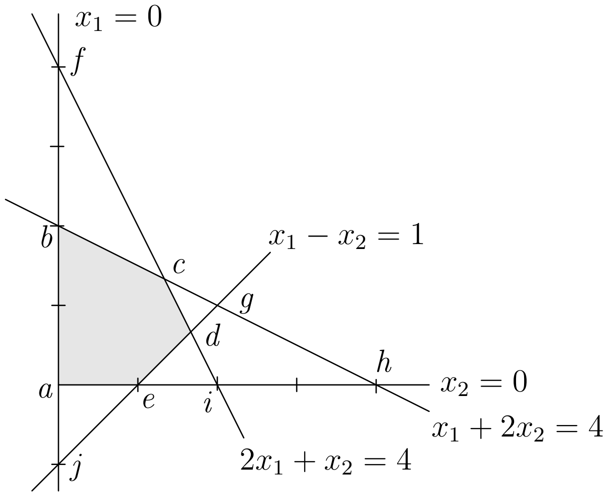
3. Introduce slack variables x_3, x_4, x_5 and extend c and A to rewrite P as

$$\max\ c_e^T x_e \quad \text{subject to } A_e x_e = b,\ x_e \geq 0.$$

4. Define a basic solution to a set of linear equations. How many basic solutions are there to the equation $A_e x_e = b$? Define a basic feasible solution. How many of these are there? Are all the basic solutions non-degenerate?

5. Write down the values of the variables $x_1, \ldots, x_5$ and of the objective function for all basic solutions to the equation $A_e x = b$ of question 3. Mark these on your diagram. Which are the basic feasible solutions?

[*The basic solutions correspond to the* $10 = \binom{5}{2}$ *intersections of the five lines defining the feasible set. Since no vertex is the intersection of more than two lines no basic solution is degenerate. Each line is defined by* $x_i = 0$ *for some* i.



	x_1	x_2	x_3	x_4	x_5	f
a	0	0	4	4	1	0
b	0	2	2	0	3	2
c	$\frac{4}{3}$	$\frac{4}{3}$	0	0	1	$\frac{8}{3}$
d	$\frac{5}{3}$	$\frac{2}{3}$	0	1	0	$\frac{7}{3}$
e	1	0	2	3	0	1
f	0	4	0	-4	5	4
g	2	1	-1	0	0	3
h	4	0	-4	0	-3	4
i	2	0	0	2	-1	2
j	0	-1	5	6	0	-1

There are 5 basic feasible solutions, a–e.]

6. Introduce slack variables λ_4 and λ_5 into the problem D from question 2. Write down the value of the variables $\lambda_1, \ldots \lambda_5$ and of the objective function for each basic solution of D. Which are the basic feasible solutions of D?

[*The basic feasible solutions are c, f, g, h.*

	λ_4	λ_5	λ_1	λ_2	λ_3	f
a	-1	-1	0	0	0	0
b	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	2
c	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{8}{3}$
d	0	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{7}{3}$
e	0	-2	0	0	1	1
f	1	0	1	0	0	4
g	0	0	0	$\frac{2}{3}$	$\frac{1}{3}$	3
h	0	1	0	1	0	4
i	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	2
j	-2	0	0	0	-1	-1

]

7. Let $z_1 = x_3$, $z_2 = x_4$, $z_3 = x_5$ be the slack variables for P and $v_1 = \lambda_4$, $v_2 = \lambda_5$ be the slack variables for D. Show that for each basic solution x of P there is exactly one basic solution λ of D such that (i) $c_e^T x_e = b^T \lambda$ (same objective value) and (ii) $\lambda_i z_i = 0$, $i = 1, 2, 3$ and $x_j v_j = 0$, $j = 1, 2$ (complementary slackness).

For how many pairs $\{x_e, \lambda\}$ is x_e feasible for P and λ feasible for D?

[*At this stage this is just a matter for observation. Only one pair is feasible for both problems; the pair of optimal solutions. Note i and j are infeasible for both problems.*]

8. Solve problem P using the simplex algorithm starting with initial basic feasible solution $x_1 = x_2 = 0$. Try both choices of the variable to put into the basis on the first step. Compare the objective rows of the various tableaux generated with appropriate basic solutions to problem D? What do you observe?

[*The x_1 way takes three steps (a–e–d–c) and the x_2 way takes two steps (a–b–c), as should be obvious by observing how many vertices of the feasible region one needs to traverse to reach the optimum.*
The students now have a tableau for each of the five vertices of the feasible region and should observe that the objective row is given by $(-\lambda_4, -\lambda_5, -\lambda_1, -\lambda_2, -\lambda_3)$ where λ is the corresponding solution of the dual. For example, the final tableau (at vertex c) is:

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{4}{3}$
x_5	0	0	-1	1	1	1
x_1	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{8}{3}$

]

Linear Programming: simplex, two-phase algorithm (Lectures 3–4)

LP1. Use the simplex algorithm to solve

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 3x_3 \\ \text{subject to} & 2x_1 + x_2 + x_3 \le 2 \\ & x_1 + 2x_2 + 3x_3 \le 5 \\ & 2x_1 + 2x_2 + x_3 \le 6 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

Each row of the final tableau is the sum of scalar multiples of the rows of the initial tableau. Explain how to determine the scalar multipliers directly from the final tableau.

[*Optimal $f = \frac{27}{5}$ at $(\frac{1}{5}, 0, \frac{8}{5}, 0, 0, 4)$. Look in the columns in the final tableau which contained the identity matrix in the initial tableau.*

	x_1	x_2	x_3	z_1	z_2	z_3	
x_1	1	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$
x_3	0	$\frac{3}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$
z_3	0	1	0	-1	0	1	4
	0	$-\frac{7}{5}$	0	$-\frac{6}{5}$	$-\frac{3}{5}$	0	$-\frac{27}{5}$

]

LP2. Define P(ϵ) to be the LP problem obtained by replacing the vector $b = (4, 4, 1)^T$ in P (question 1) by the perturbed vector $b(\epsilon) = (4 + \epsilon_1, 4 + \epsilon_2, 1 + \epsilon_3)^T$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, for the optimal value for P(ϵ) when the ϵ_i are small. If $\epsilon_2 = \epsilon_3 = 0$, for what range of ϵ_1 values does the formula hold?

[*The change in the optimal value is* $\sum_{i=1}^{3} \lambda_i \epsilon_i$ *where* λ *is the dual solution at the optimum. Hence* $\frac{1}{3}\epsilon_1 + \frac{1}{3}\epsilon_2$ *in this case. For* $\epsilon_2 = \epsilon_3 = 0$*, the formula is valid when* $-2 \le \epsilon_1 \le 1$ *(by looking at the diagram). The explanations for this formula are (in order of their appearance in lectures): (1) The entries in the final objective row, in the columns corresponding to the basic variables in the initial tableau, tell you how many multiples of each initial row were added to the original objective row as the algorithm progressed to the optimum. Thus, since the values of b do not affect the multipliers used in the pivot operations, we have a formula for the optimal value as a function of the* b_i*, which will be valid provided the sequence of pivots remains the same. (2) The entries in the final objective row are* -1 *times the dual variables. If the change in b is small then the optimum occurs with the same primal variables basic, and hence the dual optimum will occur with the same dual basis. But the dual feasible region doesn't depend on b, so the values of the optimal dual variables* λ_i *don't change. But the primal optimal value = dual optimal value =* $b^T\lambda$*, so the dependence on b follows.*]

LP3. Apply the simplex algorithm to

$$\begin{array}{lrcl} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \le & 4 \\ & -x_1 + x_2 & \le & 3 \\ & x_1, x_2 & \ge & 0. \end{array}$$

Explain what happens with the use of a diagram.

[*After one pivot on* a_{22} *we have the tableau shown. In column 1 all entries are negative and this indicates that* x_1 *can be increased without limit. Therefore the problem is unbounded*]

	x_1	x_2	z_1	z_2	
z_1	-1	0	1	2	10
x_2	-1	1	0	1	3
	4	0	0	-3	-9

LP4. Use the two-phase algorithm to solve:

$$\begin{array}{lrcl} \text{maximize} & -2x_1 - 2x_2 & & \\ \text{subject to} & 2x_1 - 2x_2 & \le & 1 \\ & 5x_1 + 3x_2 & \ge & 3 \\ & x_1, x_2 & \ge & 0. \end{array}$$

[Hint: You should get $x_1 = \frac{9}{16}$, $x_2 = \frac{1}{16}$. Note that it is possible to choose the first pivot column so that phase I last only one step. But this requires a different choice of pivot column than the one specified by the usual rule-of-thumb.]

[*I did not show them the big M method; instead I carried the phase II objective function along as an extra row in phase I. Also, I used the tableau to make the initial transformation of the phase I objective to get it in terms of non-basic variables. Solution is* $-\frac{5}{4}$, $x_1 = \frac{9}{16}$,

$x_2 = \frac{1}{16}$. *Note that if you consider just phase I of this problem it is* ***not*** *best to choose the variable with the largest positive coefficient to enter the basis. Both choices take the same number of steps (2) overall but phase I takes 1 step if x_2 is chosen to enter the basis and 2 steps if x_1 is chosen.*]

LP5. Use the two-phase algorithm to solve:

$$\begin{array}{lrcl} \text{minimize} & 13x_1 + 5x_2 - 12x_3 & & \\ \text{subject to} & 2x_1 + x_2 + 2x_3 & \leq & 5 \\ & 3x_1 + 3x_2 + x_3 & \geq & 7 \\ & x_1 + 5x_2 + 4x_3 & = & 10 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

[$x_1 = x_2 = x_3 = 1$. *It helps to choose pivots so that the fractions don't get out of control.*]

Lagrangian methods and duality (Lectures 5–6)

L1. Minimize each of the following functions in the region specified.

(a) $3x$ in $\{x : x \geq 0\}$; (b) $x^2 - 2x + 3$ in $\{x : x \geq 0\}$; (c) $x^2 + 2x + 3$ in $\{x : x \geq 0\}$.

For each of the following functions specify the set Y of λ values for which the function has a finite minimum in the region specified, and for each $\lambda \in Y$ find the minimum value and (all) optimal x.

(d) λx subject to $x \geq 0$; (e) λx subject to $x \in \mathbb{R}$; (f) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \in \mathbb{R}$;

(g) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \geq 0$; (h) $(\lambda_1 - \lambda_2)x$ subject to $0 \leq x \leq M$.

[*This question is intended to get students to see that these minimizations are easy before they get entangled with Lagrangian methods. Answers: (d) $\lambda \geq 0$, 0, $x = 0$ if $\lambda > 0$ and any $x \geq 0$ if $\lambda = 0$; (e) $\lambda = 0$, 0, any x; (f) $\lambda_1 > 0$ (minimum by differentiation) or $\lambda_1 = \lambda_2 = 0$ (minimum 0 any x); (g) $\lambda_1 > 0$ (minimum at $x = -\lambda_2/2\lambda_1$ if $\lambda_2 \leq 0$ and at $x = 0$ if $\lambda_2 > 0$) or $\lambda_1 = 0$, $\lambda_2 \geq 0$ (min at $x = 0$ unless $\lambda_2 = 0$ when any x will do); (h) (useful when we get to network problems) any λ's, minimum at $x = 0$ if $\lambda_1 > \lambda_2$, $x = M$ if $\lambda_1 < \lambda_2$ and any $0 \leq x \leq M$ if $\lambda_1 = \lambda_2$.*]

L2. Use the Lagrangian Sufficiency Theorem to minimize ${x_1}^2 + 2{x_2}^2$ subject to $x_1 + 3x_2 = b$.

[$x_1 = \frac{2}{11}b$, $x_2 = \frac{3}{11}b$, $\lambda = \frac{4}{11}b$, $f = \frac{2}{11}b^2$. *Note $df/db = \lambda$.*]

L3. (a) Use the Lagrangian Sufficiency Theorem to solve:

$$\begin{array}{lrcl} \text{maximize} & x + 2y + z & & \\ \text{subject to} & x^2 + y^2 + z^2 & \leq & 1 \\ & x, y, z & \geq & 0. \end{array}$$

(b) Find a point in $\mathbb{R}^3$ at which a plane parallel to $x + 2y + z = 0$ is tangent to the unit sphere.

[$f = \sqrt{6}$. $\lambda = -\sqrt{3/2}$, $2x = y = 2z = -1/\lambda$.]

L4. Minimize $f = \sum v_i x_i^{-1}$ in $x \geq 0$ subject to $\sum a_i x_i \leq b$ where $a_i, v_i > 0$ for all i and $b > 0$. [In this example f is the variance of an estimate derived from a stratified sample survey subject to a cost constraint; x_i is the size of the sample for the i^{th} stratum, the a_i and v_i are measures of sampling cost and of variability for this stratum.] Check that the change in the minimal variance f for a small change δb in available resources is $\lambda\,\delta b$ where λ is the Lagrange multiplier.

[*Slack variable term* $\Rightarrow \lambda \leq 0$. *Write* $k = \sum \sqrt{a_i v_i}$. *Solution is* $f = k^2 b^{-1}$ *at* $x_i = bk^{-1}\sqrt{v_i a_i^{-1}}$, *with* $\lambda = -k^2 b^{-2}$. *Note that we know the slack variable* $z = 0$ *once we see that* $\lambda = 0$ *is not a satisfactory choice.*]

Linear programming and duality (Lecture 6)

D1. Write down the Lagrangian for each of the following problems. In each case find the set Y of λ values for which the Lagrangian has a finite minimum (subject to the appropriate regional constraint), calculate the minimum $L(\lambda)$ for each $\lambda \in Y$, and write down the dual problem. In each case, write down the conditions for primal and dual feasibility and any additional conditions (the complementary slackness conditions) needed for optimality.

(a) min $c^T x$ subject to $Ax \leq b$, $x \geq 0$;   (b) min $c^T x$ subject to $Ax = b$, $x \geq 0$.

[*They will have seen this for (a) and (b) in the lectures, and for (a) in the notes.*]

D2. For each of the problems in the previous question verify that the dual of the dual problem is the primal problem.

Suppose that an LP problem P is written in the two equivalent forms

$$\min\ c^T x \quad \text{subject to} \quad Ax \leq b, \quad x \geq 0,$$

where A is an $m \times n$ matrix, $c, x \in \mathbb{R}^n$ and

$$\min\ c_e^T x_e \quad \text{subject to} \quad A_e x_e = b, \quad x_e \geq 0,$$

where, after addition of slack variables and extension of the matrix A and vector c in the appropriate way A_e is $m \times (n+m)$, and $c_e, x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of problem P, and show that the dual problems are equivalent to each other.

[*Dual of first version is max* $b^T\lambda$ *s.t.* $A^T\lambda \leq c$, $\lambda \leq 0$. *Dual of second is max* $b^T\lambda$ *s.t.* $A_e^T\lambda \leq c_e$. *But* $A_e = (A|I)$ *and* $c_e = (c, 0)$ *so the two duals are equivalent.*]

D3. Consider the problem

$$\begin{array}{lrcl} \text{minimize} & 2x_1+3x_2+5x_3+2x_4+3x_5 & & \\ \text{subject to} & x_1+\ x_2+2x_3+\ x_4+3x_5 & \geq & 4 \\ & 2x_1-2x_2+3x_3+\ x_4+\ x_5 & \geq & 3 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq & 0. \end{array}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.

[*Dual problems are not just sent to try us. This is easier than using the two-phase method. Dual has max* $f = 5$ *with* $\lambda_1 = \frac{4}{5}$, $\lambda_2 = \frac{3}{5}$, *with constraints 1 and 5 tight. Hence* x_1 *and* x_5 *are basic, so* $x_1 = x_5 = 1$.]

Algebra of linear programming (Lectures 7–8)

LP6. Consider the three equations in 6 unknowns given by $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Choose B $= \{1, 3, 6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_3, x_6)^T$, $x_N = (x_2, x_4, x_5)^T$ and the matrices A_B and A_N are constructed appropriately.

Now, write $c^T x = c_B^T x_B + c_N^T x_N$ and hence write $c^T x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate the variables x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. For $c = (3, 1, 3, 0, 0, 0)^T$ write $c^T x$ in terms of the non-basic variables. Prove directly from the formula for $c^T x$ that the basic solution you have computed is optimal for the problem maximize $c^T x$ subject to $x \geq 0$, $Ax = b$.

Compare your answer with your answer to question LP1 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$. Why is it not fair to say that the simplex algorithm is just a complicated way to invert a matrix?

[*An exercise to help with understanding the bit of the course which deals with the algebra of the simplex algorithm. In addition to inverting the matrix* A_B *(and multiplying it into* A_N *and* b*) the simplex algorithm also tells you how to choose B.*]

LP7. (Degeneracy.) Take problem P from question 1 and add the additional constraint $x_1 + 3x_2 \leq 6$. Use the simplex algorithm putting x_2 into the basis at the first stage. Try each of the possibilities for the variable leaving the basis. Explain, with a diagram, what happens.

[*You get the choice of taking* z_2 *or* z_4 *out of the basis at the first step. The other will remain basic but become 0. Just keep obeying the simplex rules and the answer comes out.*

One way to understand what happens when you choose z_4 to leave the basis is to imagine a small perturbation to the fourth (additional) constraint so that it cuts the top corner off the feasible region. Note that degeneracy can cause cycling, but it usually doesn't except in very contrived circumstances. Bazaraa, Jarvis & Sherali, Linear Programming and Network Flows, have a good discussion (pgs 164–187), including an example of a problem with $m = 3$, $n = 7$ and a cycle of length 6; they quote a result that the smallest problem which will allow cycling has $m = 2$, $n = 6$ and a cycle of length 6.]

LP8. In the previous question the additional constraint was redundant (it did not alter the feasible set). Can degeneracy occur without redundant equations?

[*Yes, for example imagine a pyramid with four or more faces meeting at the vertex. You might conjecture that in 2 dimensions, a degenerate basis implies there is a redundant constraint. But even this is not true. Consider $(x, y) \geq 0$, $x + y \leq 1$, $x \geq 1$. The only feasible point is $(x, y) = (1, 0)$ with slack and surplus variables both 0. Thus, each of the possible feasible bases is degenerate, but no contraint is redundant.*]

LP9. Show that introducing slack variables in a LP does not change the extreme points of the feasible set by proving (using the definition of an extreme point) that x is an extreme point of $\{x : x \geq 0, Ax \leq b\}$ if and only if $\binom{x}{z}$ is an extreme point of $\{\binom{x}{z} : x \geq 0, z \geq 0, Ax + z = b\}$.

[*An exercise in the use of the definition of extreme point.*]

Game theory (Lecture 9)

G1. Give sufficient conditions for strategies $\mathbf{p}$ and $\mathbf{q}$ to be optimal for a two-person zero-sum game with pay-off matrix A and value v.

Two players fight a duel: they face each other $2n + 1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends. Otherwise both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the i^{th} pace forward ($i = 0, 1, \ldots, n$) is i/n. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired.

Show that, if $n = 4$, the strategy 'shoot after taking two steps' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11})$.]

[*Slightly reworded from the old tripos question, but it means the same. First part: $\mathbf{p}^T A \geq v$, $A\mathbf{q} \leq v$, and $\mathbf{p}^T A\mathbf{q} = v$ where p and q are probability vectors and the inequalities hold in each component. Second part: Clearly nobody shoots before the first step so the possibilities are $i = 1, \ldots, n$. Let*

$$a_{ij} = E\Big(\textit{payoff to player I} \mid \textit{Player I fires at step } i \textit{ and Player 2 fires at step } j\Big).$$

For $i < j$, $E = (i-j)/n + ij/n^2$, and the game is symmetric with value 0. Thus,

$$A_4 = \frac{1}{16}\begin{pmatrix} 0 & -2 & -5 & -8 \\ 2 & 0 & 2 & 0 \\ 5 & -2 & 0 & 8 \\ 8 & 0 & -8 & 0 \end{pmatrix} \quad \text{and} \quad A_5 = \frac{1}{25}\begin{pmatrix} 0 & -3 & -7 & -11 & -15 \\ 3 & 0 & 1 & -2 & -5 \\ 7 & -1 & 0 & 7 & 5 \\ 11 & 2 & -7 & 0 & 15 \\ 15 & 5 & -5 & -15 & 0 \end{pmatrix}.$$

The optimality conditions may be checked with these matrices and the suggested solutions. Alternatively, for a minimal answer just observe that for $n = 4$ the point $(2,2)$ is a saddle-point for the game and that for $n = 5$ there can be no optimal pure strategy since there is no non-positive column (or non-negative row).]

G2. Optimal strategies are not always unique as illustrated by the following payoff matrix,

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}.$$

Find all optimal strategies.

[*$v = 0$ since the game is symmetric. Any strategy $(1-d, 0, 0, d)$, with $\frac{2}{5} \le d \le \frac{3}{7}$ is optimal.*]

G3. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n. [Hint: Guess a solution and show that it is optimal.]

[*Mixed strategy with uniform probabilities $1/n$ for both players satisfies optimality conditions for given v.*]

G4. Find optimal strategies for both players, and the value of the game, for the game with payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

[You may like to try to compare the effort required to solve this by (a) seeking strategies and a value which satisfy the optimality conditions (b) direct solution of Player I's original minimax problem and (c) using the simplex method on one of the players problems after transforming it as suggested in the lectures.]

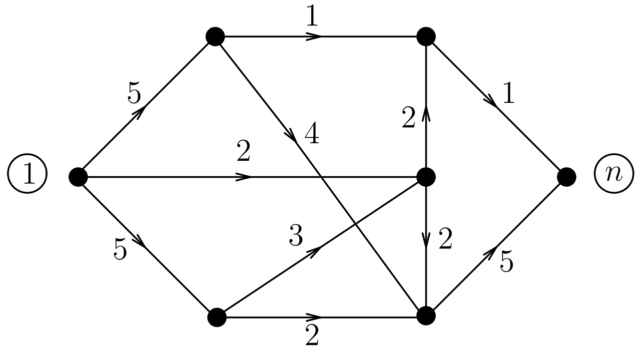
[*The solution is $\mathbf{p} = (\frac{1}{4}, \frac{3}{4})^T$, $\mathbf{q} = (\frac{1}{2}, \frac{1}{2})^T$, $v = \frac{5}{2}$. For this problem it is straightforward to solve either Player I or Players II's original problem. For example, Player I's problem is to find a strategy $(p, 1-p)$ to $\max\{\min(3-2p, 2+2p)\}$ in $0 \le p \le 1$. This has optimum $\frac{5}{2}$ at $p = \frac{1}{4}$ (by inspection). Similarly, it is easy to see the solution by attempting to satisfy the optimality conditions once you observe that $A\mathbf{q} \le v$ can only be solved for $v \ge \frac{5}{2}$ and $\mathbf{p}^T A \ge v$ can only be solved for $v \le \frac{5}{2}$. Neither of these methods is very helpful for larger problems. The method suggested in lectures is to transform Player I's problem by (i) adding* *a constant k to each a_{ij} so that each is positive (this doesn't change the solution but adds k to the value of the game), (ii) change variables to $x_i = p_i/v$ (ok since now $v > 0$), and (iii) solve $\min \frac{1}{v} = \sum x_i$ subject to $\sum a_{ij} x_i \geq 1$ for each j and $x_i \geq 0$ for each i. Step (i) is unnecessary here. The problem to solve by simplex is thus*

$$\min\ x_1 + x_2 \quad \textit{subject to} \quad x_1 + 3x_2 \geq 1, \quad 4x_1 + 2x_2 \geq 1, \quad x_1, x_2 \geq 0.$$

This has solution (after use of the two-phase algorithm, or solve the dual) $x_1 = \frac{1}{10}$, $x_2 = \frac{3}{10}$ which after appropriate rescaling gives the p_i.]

Max-flow/min-cut problems (Lecture 10)

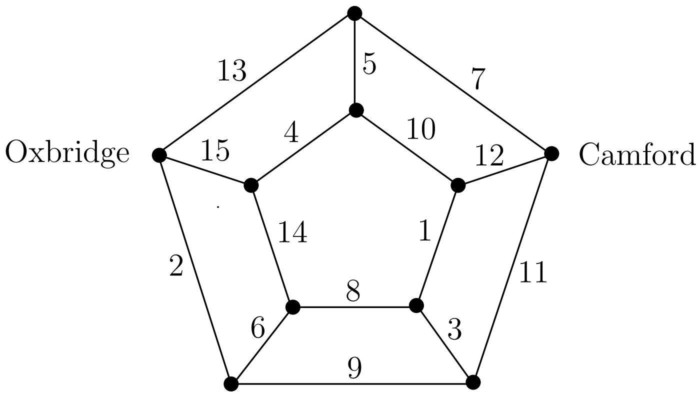
N1. Show that the maximal value of a flow through a network with a source at node 1 and a sink at node n equals the minimum cut capacity. Find a 'maximal flow' and 'minimal cut' for the network below.



[*Bookwork covered in lectures. Max flow 6 with cut separating sink n from other nodes.*]

N2. Devise rules for a version of the Ford-Fulkerson algorithm which works with undirected arcs.

As a consequence of drought, an emergency water supply must be pumped from Oxbridge to Camford along the network of pipes shown in the figure. The numbers against the pipes show their maximal capacities, and each pipe may be used in either direction. Find the maximal flow and prove that it is maximal.

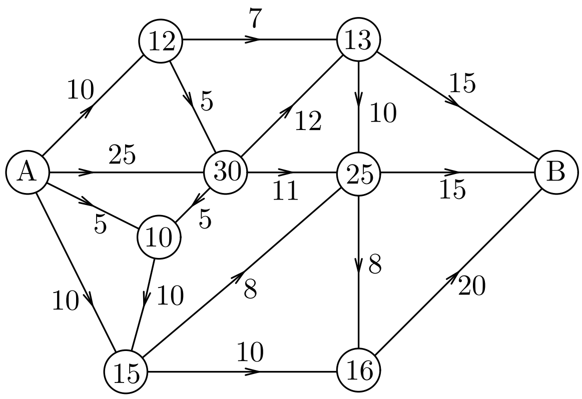


[*Optimum is 28. Proof of maximality is by exhibiting a cut with the same value (e.g., the cut that crosses (from top to bottom) arcs with capacities 7,5,4,1,11). For the algorithm, either replace each undirected arc by two directed arcs and use the FF algorithm in the normal way. Or, write* $|x_{ij}| \leq c_{ij}$ *for the capacity constraints where each arc is labelled only once (from node i to node j) even though flow is allowed in both directions. With the source at node 1 and sink at node n, define the set S of nodes inductively by (i)* $1 \in S$, *(ii) if* $i \in S$ *and* $x_{ij} < c_{ij}$ *then* $j \in S$, *(iii) if* $j \in S$ *and* $x_{ij} > -c_{ij}$ *then* $i \in S$, *until* $n \in S$ *or no more nodes can be added to S. If* $n \in S$ *then there is a path from 1 to n along which the flow should be increased by the minimum of* $min[c_{ij} - x_{ij}]$ *over arcs i to j in the path, and* $min[c_{ij} + x_{ij}]$ *over arcs j to i in the path.*]

N3. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal flow problems?

The road network between two towns A and B is represented in the diagram below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.

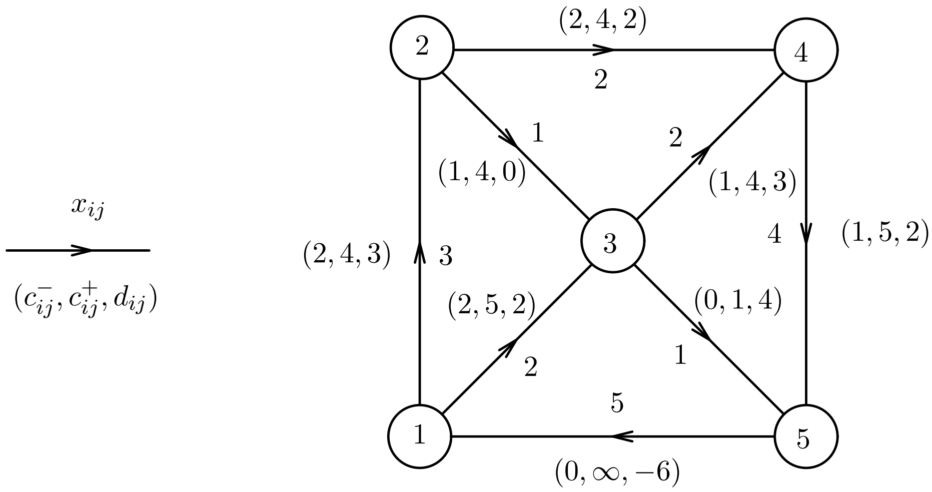
The Minister of Transport intends to build a by-pass round one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if her intent is to increase the maximal flow from A to B as much as possible? What would the new maximal flow be?



[*Split each node into two, one receiving incoming flows, the other generating the outgoing flows, and joined by an arc with capacity equal to the node capacity. Example has max flow 39. Best by-pass is the village at top right (capacity 13), increasing the max flow to 45.*]

Minimal cost circulations (Lecture 11)

N4. By finding a suitable potential on the nodes (i.e., a set of suitable node numbers) of the network show that the flow illustrated below is a minimal cost circulation. [Each arc is labelled with x_{ij} and with a triple of numbers giving the constants $(c_{ij}^-, c_{ij}^+, d_{ij})$ for that arc.]



[*Potential $\lambda_1 = -6 + c$, $\lambda_2 = -3 + c$, $\lambda_3 = -5 + c$, $\lambda_4 = -2 + c$, $\lambda_5 = 0 + c$ will do for any c. The method of solution it to set one node number to zero arbitrarily, to then solve $d_{ij} - \lambda_j + \lambda_i = 0$ on those arcs where the flow $c_{ij}^- < x_{ij} < c_{ij}^+$ (there will be sufficient such arcs in non-degenerate problems to create a spanning tree), and then to check the optimality conditions on the remaining arcs. The transportation algorithm is a special case.*]

N5. Consider the problem of assigning lecturers $L_1, \dots, L_n$ to courses $C_1, \dots, C_n$ so as to minimize student dissatisfaction. The dissatisfaction felt by students if lecturer L_i gives course C_j is d_{ij}, and each lecturer must give exactly one course. Show how to state this problem as the problem of minimizing the cost of a circulation in a network. (Can you be sure that your network problem has an optimal flow of the appropriate kind?)

For the example of 3 lecturers and 3 courses with dissatisfaction matrix

$$\begin{pmatrix} 6 & 3 & 1 \\ 5 & 4 & 3 \\ 8 & 3 & 2 \end{pmatrix}$$

find an optimal flow through the appropriate network (by guessing) and compute node numbers for each node so that the optimality conditions are satisfied.

[*Various networks will do. Some argument is needed that there will be an integer solution amongst the optimal solutions to the flow problem; one argument, given an algorithm for solving the problem that depends — as they all do — on increasing flows by the maximum permitted amount around circuits, is that there is an obvious integer initial feasible flow and that all steps of the algorithm will preserve the integer nature of the flow.*]

Transportation algorithm (Lecture 12)

N6. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 18 tons respectively. The demands for candy-floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (ij) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the associated minimal cost.

[*If the algorithm is started by the NW rule this example requires three steps before reaching the optimum. Answer 505. To aid clarity it is useful to label potentials with* λ*'s on the source nodes and* μ*'s on the destination nodes, so we require* $d_{ij} - \lambda_i + \mu_j = 0$ *on arcs with non-zero flow. Students should not be discouraged from guessing better initial solutions than those given by the NW rule. The node numbers and optimal allocations are shown in the final tableau below. Everywhere that* $x_{ij} = 0$ *we have* $d_{ij} - \lambda_i + \mu_j \geq 0$.

$\lambda_i \setminus \mu_j$	0	-4	3	
0	[7]	20 [4]	[9]	20
8	21 [8]	14 [12]	7 [5]	42
3	18 [3]	[11]	[7]	18
	39	34	7	

]

Further Examples

Linear programming, simplex algorithm (Lectures 3–4)

F1. Consider the problem $\min \sum_{i=0}^{n} |x_i|$ subject to $Ax \leq b$ where A is $m \times n$ and $b \in \mathbb{R}^m$. Show how to convert the problem into a standard LP problem suitable for solution by the simplex algorithm. What happens if you replace min by max?

[*Replace* x_i *in all constraints by* $y_i - z_i$*, where* $y_i, z_i \geq 0$*, and take the objective function to be* $\sum_i (y_i + z_i)$*. It is clear that at the optimum at most one of* y_i *and* z_i *will non-zero and they will correspond respectively to the positive and negative parts of* x_i*. So* $y_i + z_i = |x_i|$*. If min is replaced by max there is no finite optimum.*]

F2. Maximize $3x_1 + 6x_2 - x_3$ subject to $x_i \geq 0$ and

$$\begin{aligned} -x_1 + 4x_2 - x_3 &\leq 2 \\ 2x_1 + x_2 + x_3 &\leq 5 \\ -x_1 + x_2 + 2x_3 &\leq 1. \end{aligned}$$

Write down the dual problem and its solution. Now replace the second constraint by $2x_1 + x_2 + x_3 \le 5 + t$. Find the new maximum for small values of t. For what range of t values is this solution valid?

[*The final tableau is as shown. Hence the dual solution is* $\lambda = (1, 2, 0)$.

	x_1	x_2	x_3	z_1	z_2	z_3	
x_2	0	1	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	0	1
x_1	1	0	$\frac{5}{9}$	$-\frac{1}{9}$	$\frac{4}{9}$	0	2
z_3	0	0	$\frac{8}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1	2
	0	0	-2	-1	-2	0	-12

The new maximum is $12 + 2t$ *and this holds provided* $1 + \frac{1}{9}t \ge 0$, $2 + \frac{4}{9}t \ge 0$ *and* $2 + \frac{1}{3}t \ge 0$, *i.e., for* $t \ge -\frac{9}{2}$.]

F3. Use the simplex algorithm to solve

$$\begin{array}{lrcl} \text{minimize} & 5x_1 - 3x_2 & = & f \\ & 2x_1 - x_2 + 3x_3 & \le & 4 \\ & x_1 + x_2 + 2x_3 & \le & 5 \\ & 2x_1 - x_2 + x_3 & \le & 1 \\ & x_1, x_2, x_3 & \ge & 0. \end{array}$$

Write down the dual problem, and solve it by inspection of the final tableau for the primal. If the constraints in the right hand side of the above problem are changed to $4 + \varepsilon_1$, $5 + \varepsilon_2$, $1 + \varepsilon_3$, respectively, for small $\varepsilon_1, \varepsilon_2, \varepsilon_3$, by how much does the optimum value of f change?

[*The final tableau is as shown. Hence the dual solution is* $\lambda = (0, -3, 0)$.

	x_1	x_2	x_3	z_1	z_2	z_3	
x_4	3	0	5	1	1	0	9
x_2	1	1	2	0	1	0	5
z_3	3	0	3	0	1	1	6
	8	0	6	0	3	0	15

So with small changes the optimum changes to $-15 - 3\epsilon_2$.]

Lagrangian sufficiency theorem (Lectures 5–8)

F4. Define the Lagrangian $L(x; \lambda)$ for the problem: minimize $f(x)$ in $x \in X$ subject to $g(x) = b$, where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $b \in \mathbb{R}^m$, $X \subset \mathbb{R}^n$.

Show that if there exist $\overline{x} \in X$ and $\overline{\lambda} \in \mathbb{R}^m$ such that

$$L(\overline{x}; \overline{\lambda}) \le L(x; \overline{\lambda}) \quad \text{for all} \quad x \in X$$

and $g(\overline{x}) = b$ then $\overline{x}$ is optimal for the above problem.

Use the above result to minimize $(v_1/x_1) + (v_2/x_2)$ subject to $x_1 c_1 + x_2 c_2 = c$, $x_1 \ge a_1$, $x_2 \ge a_2$, where c_1, c_2 and a_1, a_2, v_1, v_2 and c are given positive constants, and $c \ge a_1 c_1 + a_2 c_2$.

[*The Lagrangian is minimized by* $x_i = \max\left\{a_i, \sqrt{-v_i/\lambda c}\right\}$, *for* $\lambda < 0$. *Note that* $x_1(\lambda)c_1 + x_2(\lambda)c_2$ *increases from* $a_1c_1 + a_2c_2$ *to* ∞ *as* λ *increases from* $-\infty$ *to* 0. *So the condition* $c \geq a_1c_1 + a_2c_2$ *ensures that there exists a* λ *such that* $x_1(\lambda)c_1 + x_2(\lambda)c_2 = c$.]

F5. A gambler at a horse race has an amount b to bet. The gambler assesses p_i, the probability that horse i will win, and knows that s_i has been bet on horse i by others, for $i = 1, 2, \dots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gamblers optimal strategy is to choose $(x_1, x_2, \dots, x_n)$ to

$$\max \sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = b, \quad x_1, x_2, \dots, x_n \geq 0\,,$$

where x_i is the amount the gambler bets on horse i.

Find the form of the gamblers optimal strategy. Deduce that if b is small enough, the optimal strategy is to bet only on the horses for which the ratio p_i/s_i is maximal.

[*The Lagrangian is maximized by* $x_i = \max\left\{0, s_i\left(\sqrt{p_i/s_i\lambda} - 1\right)\right\}$, *for* $\lambda > 0$. *As* λ *increases from* 0 *the value of* $\sum_i x_i(\lambda)$ *decreases from* ∞ *to* 0. *Thus if* b *is small,* λ *will be large, and when* λ *is large enough the only* x_i *that will be non-zero is one corresponding to the maximal ration of* p_i/s_i.]

F6. Two sets of n positive real numbers $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ are given, where the numbers in each set sum to 1 (i.e., they are probability distributions). It is desired to find numbers $x_1, \dots, x_n$, where $0 \leq x_i \leq 1$, such that for a given number α, $(0 < \alpha < 1)$ the quantity

$$\beta = 1 - (x_1 q_1 + \cdots + x_n q_n)$$

is minimized, subject to

$$x_1 p_1 + \cdots + x_n p_n = \alpha\,.$$

Write down the Lagrangian for this problem, and show that the minimization is achieved if, for a certain value of λ,

$$\begin{aligned} x_i &= 1 \quad \text{if} \quad q_i > \lambda p_i \\ x_i &= 0 \quad \text{if} \quad q_i < \lambda p_i \\ x_i &= \phi \quad \text{if} \quad q_i = \lambda p_i \end{aligned}$$

for a value of ϕ such that $0 < \phi \leq 1$.

[**Hint**. Without loss of generality, suppose that $q_1/p_1, \dots, q_n/p_n$ are in increasing order of magnitude. IB students should note that this is essentially a proof of the Neyman-Pearson lemma and should be able to give an interpretation to the numbers x_i.]

F7. Show how to solve the problem

$$\min \sum_{i=1}^{n} \frac{1}{(a_i + x_i)} \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = b, \quad x_i \geq 0 \ \ (i = 1, \dots, n)$$

where $a_i > 0$, $i = 1, \dots, n$ and $b > 0$.

F8. A certain lecturer eats only nuts and beans. His pathetic daily salary is $I > 0$. The quality of his lectures is given by the real number $x_1^{a_1} x_2^{a_2}$ where x_1 is his daily consumption of nuts, x_2 is his daily consumption of beans, and the a_i are positive constants, $a_1 + a_2 = 1$. The price of nuts and beans are p_1 and p_2 respectively ($p_i > 0$). The lecturer, for reasons known best to himself, wishes to give lectures that are of as high a quality as possible, without spending more than his income. Unfortunately, the lecturer is too weak to solve his own optimization problem. Please do it for him.

Despite the lecturer's best efforts, the students still find his lectures pretty awful. They decide to make a collection and succeed in raising the tiny (even in comparison with I) amount δI. The sum having been handed over with due ceremony, how much of an increase in the quality of their lectures can the students expect?

[*The Lagrangian is* $L = x_1^{a_1} x_2^{a_2} - \lambda(p_1 x_1 + p_2 x_2 - I)$. *This is maximized, for* $\lambda > 0$, *where* $x_i = a_i/p_i\lambda$. *(Check that the second derivative matrix is negative semi-definite.) To satisfy the constraint, we take* $\lambda = 1/I$ *and hence* $x_i = a_i I/p_i$. *The increase in the quality of lectures is given by* $\lambda \delta I = I^{-1}\delta I$.]

F9. Consider the problem: minimize x^2 subject to $g(x) \geq b$. Show that it is possible to use the Lagrangian Sufficiency Theorem to solve the problem when (a) $g(x) = x$ and $b = 2$, but that this is not possible for (b) $g(x) = x^3$ and $b = 8$. Explain why the Lagrangian method works in case (a) but not in case (b).

[*In case (a) the Lagrangian is* $L = x^2 - \lambda(x - z - 2)$ *and this is minimized at* $x = 2$, $z = 0$, *for* $\lambda = 4$. *However, for (b), the Lagrangian is* $L = x^2 - \lambda(x^3 - z - 8)$ *and this is minimized either at* $x = \infty$, $z = 0$ *if* $\lambda > 0$, *or at* $x = 0$, $z = \infty$ *if* $\lambda \leq 0$. *If we define* $\phi(b)$ *as the maximum value of* x *subject to* $g(x) = b$, *then Lagrangian methods work if* $\phi(b)$ *has a supporting hyperplane at* b. *In case (a)* $g(x) = x$ *and* $\phi(b) = b^2$, *which is a convex function. In case (b)* $g(x) = x^3$ *and* $\phi(b) = b^{2/3}$, *which is a concave function, and so does not have a supporting hyperplane at any point.*]

Two-person zero-sum games (Lecture 9)

F10. Follow the usual steps to find the dual problem to player I's problem

$$\max\ v \qquad \text{subject to} \qquad \sum_i a_{ij} p_i \geq v \quad (\text{each } j), \quad \sum_i p_i = 1, \quad p \geq 0.$$

Show that the dual problem is player II's problem, and use the Lagrange sufficiency theorem to find sufficient conditions for a strategy vectors $\mathbf{p}$ and $\mathbf{q}$ to be optimal for the two players.

[*Bookwork; covered in lectures.*]

F11. Let the matrix (a_{ij}), $i = 1, \ldots, m$, $j = 1, \ldots, n$ define a two-person zero-sum game. Show that there exist numbers $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ with

$$p_i \geq 0, \qquad q_j \geq 0, \qquad \sum p_i = 1, \qquad \sum q_j = 1$$

such that

$$\sum_{i,j} a_{ij} p'_i q_j \le \sum_{i,j} a_{ij} p_i q_j \le \sum_{i,j} a_{ij} p_i q'_j$$

for all numbers $p'_1, \ldots, p'_m$ and $q'_1, \ldots, q'_n$ with

$$p'_i \ge 0, \qquad q'_j \ge 0, \qquad \sum p'_i = 1, \qquad \sum q'_j = 1$$

and explain the sense in which this entails the existence of optimal strategies for the game.

In the game of Undercut each player selects secretly a number from $1, 2, \ldots, N$. The numbers are then revealed, and the player with the smaller number wins £1, unless the numbers are either *adjacent*, when the player with the larger number wins £2, or *equal*, when the game is tied (with payoff zero). Show that for $N \ge 5$ it is optimal for both players to name the numbers $1, 2, 3, 4, 5$ with probabilities $(\frac{1}{16}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{16})$ respectively, and to avoid all larger numbers.

Show, using the first part of the question or otherwise, that the optimal strategy given above is the *unique* optimal strategy for this game.

[*In the first part,* $\mathbf{p}$ *and* $\mathbf{q}$ *are solutions to the usual LP. The pay-off matrix is*

$$A = \begin{pmatrix} 0 & -2 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 0 & -2 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 2 & 0 & -2 & 1 & 1 & 1 & \cdots \\ -1 & -1 & 2 & 0 & -2 & 1 & 1 & \cdots \\ -1 & -1 & -1 & 2 & 0 & -2 & 1 & \cdots \\ -1 & -1 & -1 & -1 & 2 & 0 & -2 & \\ -1 & -1 & -1 & -1 & -1 & 2 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix}.$$

From this we can check that $\mathbf{p}^T A = (0, 0, 0, 0, 0, \frac{13}{16}, 1, 1, \ldots) \ge 0$, $A\mathbf{q} \le 0$ *and* $\mathbf{p}^T A \mathbf{q} = 0$, *so the sufficient conditions for optimality are satisfied. The solution is unique because if player I uses this strategy then player II can only use numbers* 1–5. *But there is only one solution to* $A\mathbf{q} \le 0$, $\mathbf{1}^T \mathbf{q} = 1$.]

Maximal flows (Lecture 10)

F12. A network with nodes $1, \ldots, n$ has a maximal capacity $M_{ij} \ge 0$ for transfer along the directed arc from node i to node j. Describe an algorithm to compute the maximal flow from node 1 to node n in the case when all the M_{ij} are integers. Prove that the algorithm will converge to the maximal flow in a finite number of steps.

Derive the Max-flow-Min-cut Theorem, and explain how the theorem can be used to prove that a particular flow is maximal.

F13. Formulate the maximum flow through a network problem as a linear programming problem. How many variables and constraints may be needed for a problem with n nodes? Show that the dual problem has a solution in which the variables takes only two values, and explain the significance of this result.

Applications of max-flow min-cut theorem (Lecture 10)

F14. (Konig-Egervary Theorem). Consider an $m \times n$ matrix A in which every entry is either 0 or 1. Say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set. Say that a set of 1's is *independent* if no pair of 1's of the set lies in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1's equals the minimal number of lines that cover the matrix.

[*Consider a network consisting of a source and since and $m + n$ other nodes. There are m nodes corresponding to the rows, and n nodes corresponding to the columns. There is an edge of capacity* 1 *from a column node to a row node if there is a 1 in the matrix. The row nodes are connected to the source by edges of capacity ∞ and the column nodes are connected to the sink by edges of capacity ∞. The theorem follows from min cut = max flow.*]

F15. Derive the vertex form of Menger's Theorem, which states that if A and B are nodes of an undirected network then the maximum number of node-disjoint paths from A to B which can be chosen simultaneously is equal to the minimum number of nodes whose removal disconnects A and B. [Two paths from A to B are node-disjoint if the only nodes they have in common are A and B. The removal of a set of nodes S disconnects A and B if any path from A to B passes through at least one node of S.]

[*Consider the network in which each node i in the original graph is replaced by two nodes (i_1 and i_2) joined by a directed edge ($i_1 \to i_2$) of capacity 1, and each edge in the original graph (i, j) is replaced by two directed edges $(i_2 \to j_1)$ and $(j_2 \to i_1)$ of capacity ∞. In this network The node version of Menger's Theorem follows from min cut = max flow.*]

F16. Suppose that N is a network with vertices $0, 1, 2, \ldots, 2n, 2n+1$, where 0 is the source and $2n + 1$ is the sink, such that

1. (a) for each $i = 1, \ldots, n$, $(0, i)$ is an edge of capacity 1;
2. (b) for each $j = n + 1, \ldots, 2n$, $(j, 2n + 1)$ is an edge of capacity 1;
3. (c) the only other edges have capacity n and are of the form (i, j) with $i \in \{1, \ldots, n\}$, $j \in \{n + 1, \ldots, 2n\}$,

and for each subset $I \subset \{1, \ldots, n\}$ the number of distinct vertices j such that an edge (i, j) exists for some $i \in I$ is not less than $|I|$, the number of elements in I. Prove that any maximal flow in N has value n.

Hence show (the Hall 'Marriage Theorem') that if we have a set of n boys and a set of n girls, such that every subset B of the boys between them know at least $|B|$ of the girls, then they can pair off, each boy with a girl whom he knows.

[*We need to show that the minimum cut is n. It is certainly no more than n because we could simply cut through all edges of type (a). If it were less than n then it could not cut through any edge of type (c). Consider a cut that cuts exactly r edges of type (a).*

Then there is a set I of $n-r$ nodes in $\{1,\dots,n\}$ connected to 0. Therefore, by the given condition, there are at least $n-r$ nodes in $\{n+1,\dots,2n\}$ connected to 0 via the nodes in I. To separate node 0 from node $2n+1$ requires that the cut to go through at least $n-r$ edges of type (b). Thus the cut has value at least $r+(n-r)=n$.

The Marriage Theorem is an almost immediate corollary of the above. However, it is important to note that there exists a maximal flow of n in which the flow on every edge is either 0 or 1 (since we might have obtained this flow using the Ford-Fulkerson Algorithm). This integer-valued flow pairs off the boys and girls.]

Transportation algorithm (Lecture 12)

F17. What is meant by a transportation problem? Describe an algorithm for solving such a problem, illustrating your account by solving the problem with three sources and three destinations described by the table

4	3	1	10
6	10	3	8
3	5	7	8
3	9	14	

where the figures in the boxes denote transportation costs (per unit), the right-hand column denotes supplies, and the bottom row denotes demands.

[*Optimum is* 76 *with node numbers and allocations given in the tableau below.*

$\lambda_i \setminus \mu_j$	-1	-3	-1	
0	[4]	4 [3]	6 [1]	10
2	[6]	[10]	8 [3]	8
2	3 [3]	5 [5]	[7]	8
	3	9	14	

Everywhere that $x_{ij}=0$ we have $d_{ij}-\lambda_i+\mu_j \geq 0$.]

F18. A manufacturer has to supply $\{5,7,9,6\}$ units of a good in each of the next four months. He can produce up to 8 units each month on ordinary time at costs $\{1,3,4,2\}$ per unit, and up to 3 extra each month on overtime at costs $\{2,5,7,4\}$ per unit (where costs are given for each of the next four months). Storage costs are 1 per unit per month. He desires to schedule production to minimize costs over the four-month period. Formulate his problem as a transportation problem and solve it.

[Hint. Formulate this problem as one with 8 sources and 5 destinations.]

[*Optimum is* 71. *The problem formulation is clear from the tableau below, in which we show the node numbers and allocations of an optimal allocation. There are many optimal* *allocations.*

$\lambda_i \setminus \mu_j$	-1	-2	-3	-1	1	
0	5 $\boxed{1}$	3 $\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{0}$	8
1	$\boxed{2}$	3 $\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{0}$	3
1	$\boxed{\infty}$	1 $\boxed{3}$	7 $\boxed{4}$	$\boxed{5}$	$\boxed{0}$	8
1	$\boxed{\infty}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	3 $\boxed{0}$	3
1	$\boxed{\infty}$	$\boxed{\infty}$	2 $\boxed{4}$	$\boxed{5}$	6 $\boxed{0}$	8
1	$\boxed{\infty}$	$\boxed{\infty}$	$\boxed{7}$	$\boxed{8}$	3 $\boxed{0}$	3
1	$\boxed{\infty}$	$\boxed{\infty}$	$\boxed{\infty}$	6 $\boxed{2}$	2 $\boxed{0}$	8
1	$\boxed{\infty}$	$\boxed{\infty}$	$\boxed{\infty}$	$\boxed{4}$	3 $\boxed{0}$	3
	5	7	9	6	17	

This satisfies the optimality conditions that everywhere that $x_{ij} > 0$ *we have* $d_{ij} - \lambda_i + \mu_j = 0$ *and everywhere that* $x_{ij} = 0$ *we have* $d_{ij} - \lambda_i + \mu_j \geq 0$.]