Tripos Questions in Optimization IB (1985-98)

Parts of some of these questions are no longer relevant to the syllabus. You should not be concerned to do the second half of 87306, 87406, or the very last bit of 85306.

99305 Consider the zero-sum game with pay-off matrix

$$\begin{pmatrix} 0 & 1 & 6 \\ 4 & 2 & 0 \end{pmatrix}.$$

Write down sufficient conditions for the optimality of a pair of strategies for the players. Hence or otherwise determine optimal strategies for the players and the value of the game.

99315 Consider the linear optimization problem:

$$\begin{array}{lrrrl} \text{minimize} & 18y_1 & -4y_2 & +8y_3 & \\ \text{subject to} & y_1 & +y_2 & +2y_3 & \geq 1 \\ & y_1 & & -y_3 & \geq 3 \\ & y_1 & -2y_2 & +y_3 & \geq 5 \\ & & & y_1, y_2, y_3 & \geq 0. \end{array}$$

Obtain the dual and solve both problems.

99405 State and prove the max flow/min cut theorem.

99414 Electric current in a network of resistors is the flow which minimizes the rate of energy dissipation, given by

$$I(f) = \frac{1}{2} \sum_{i,j \in S} a_{ij}^{-1} f_{ij}^2.$$

Here S denotes the finite set of nodes $a_{ij} = a_{ji} \in (0, \infty)$ is the conductivity of the resistor connecting i and j and f_{ij} is the flow from node i to node j. [*Thus, in particular, for the current, at least one of f_{ij} and f_{ji} is zero.*]

Consider the problem of determining the current in the network when unit current enters at a and leaves at b, and current is conserved at all other nodes. Express this as a constrained optimization problem and write down its Lagrangian.

Suppose that we can find a unit flow f from a to b and a potential λ_{ij}, satisfying Ohm's law

$$f_{ij} = \begin{cases} a_{ij}(\lambda_j - \lambda_i) & \text{if } \lambda_j \geq \lambda_i, \\ 0 & \text{if } \lambda_j < \lambda_i. \end{cases}$$

Show that f is then the unit currrent from a to b.

The diagram illustrates a particular network of resistors, with conductivities written on each edge. Determine the unit current from a to b.

98305 Discuss carefully the following problem:

$$\text{maximise} \quad f(x) = ax^2 + bx + c \quad \text{in} \quad 0 \le x \le 10$$

where a, b, c are given constants.

98315 Consider the set S defined by

$$S = \{(x, y, z) : \quad x + y + z \le 2, \quad 3x + y + z \le 3, \quad x, y, z \ge 0\}.$$

(a) Prove that S is a convex set. (b) List the extreme points of S. (c) Find the maximum value of $x + 2y + z$ for $(x, y, z) \in S$. (d) Write down the Lagrangian for problem (c), and hence find the dual problem.

98405 Consider a network representing the problem of a supplier who has n supply depots from which goods must be shipped to m ports. Assume that there are quantities $s_1, \dots, s_n$ of the goods at depots $\{S_1, \dots, S_n\}$, and that the demands at the ports $\{D_1, \dots, D_m\}$ are given by $d_1, \dots, d_m$, where $s_1 + \dots + s_n = d_1 + \dots + d_m$.

The cost of transporting an amount of goods x_{ij} from S_i to D_j is $d_{ij}x_{ij}$, for $i = 1, \dots, n$ and $j = 1, \dots, m$.

Using the Lagrangian sufficiency theorem, which you should state carefully, and dual variables, derive necessary and sufficient conditions for x_{ij} to minimise

$$\sum_i \sum_j d_{ij}x_{ij}$$

subject to the constraints given above.

98414 Suppose a_1, a_2, a_3, b are given positive numbers. Define $\phi(b)$ by

$$\phi(b) = \max \sum_{i=1}^{3} x_i \ ,$$

where the maximum is taken over the region

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i} \le b \quad \text{and} \quad x_1, x_2, x_3 \ge 0.$$

Derive an expression for $\phi(b)$, and give a sketch to illustrate the solution.

If b is changed to $b + \delta b$, compute the change in ϕ to first order in δb.

97305 State carefully, and prove, the Lagrangian Sufficiency Theorem.

97315 State carefully the max-flow min-cut theorem. In the case where all capacity constraints are integral, comment on an integrality property of at least one optimal flow.

Let A be an $m \times n$ matrix in which each entry is wither 0 or 1. We say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set, and that a set of 1s is *independent* if no pair of 1s of the set lies in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1s equals the minimal number of lines that cover the matrix.

97405 Determine the value and optimal strategies for the two-person zero-sum game with payoff matrix

$$\begin{pmatrix} a & 2 \\ 4 & 1 \end{pmatrix}$$

for each value of a, $0 < a < \infty$. Here, the (i, j) entry of the matrix is the payoff of the row player if row i and column j are chosen.

97414 Use the two-phase simples algorithm to solve the following linear programme:

$$\begin{array}{llllllll}
\text{maximize} & 2x_1 & - & x_2 & + & 5x_3 & \\
\text{subject to} & -x_1 & + & x_2 & - & x_3 & \geq 1 \\
 & x_1 & + & 2x_2 & - & 4x_3 & \geq 1 \\
 & x_1 & + & x_2 & + & x_3 & \leq 3 \\
\text{in} & & & x_1, x_2, x_3 & & & \geq 0
\end{array}$$

96305

(a) Define the *dual* of the linear program:

$$\begin{array}{ll} \text{maximize} \quad c^\top x \quad \text{in} \quad x \geq 0 \\ \text{subject to} \quad Ax \leq b \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Show that is x is feasible for the above *primal* problem and λ is feasible for its dual then $c^\top x \leq \lambda^\top b$.

(b) Use the simplex algorithm to solve either the primal problem or the dual problem when

$$A = \begin{pmatrix} -1 & -3 & 2 \\ 2 & -6 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -1 \\ -3 \end{pmatrix},$$

and then solve the other problem.

96405

(a) State and prove the Lagrangian sufficiency theorem.

(b) Consider the following problem:

$$\text{minimize} \quad \sum_{i=1}^n a_i e^{-x_i}$$

$$\text{subject to} \quad \sum_{i=1}^n x_i = b, \quad x_1, x_2, \dots, x_n \geq 0$$

where $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $b > 0$. Show that in the optimal solution $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, and that $x_j > 0$ if and only if

$$a_1 a_2 \cdots a_j < e^b (a_j)^j.$$

95305

(a) Find the value and the optimal strategies for each player for the matrix game with payoff matrix

$$\begin{pmatrix} 1 & 0 & 4 \\ -1 & 1 & -2 \end{pmatrix}.$$

(b) Formulate the dual of the linear programming problem

$$\begin{array}{lrcll} \text{Minimize} & 7x_1 + 10x_2 + 11x_3 & & & \\ \text{subject to} & x_1 + \;\; 3x_2 + \;\; 8x_3 & \geq & 1 & \\ & x_1 + \quad x_2 + \;\; 2x_3 & \geq & 1 & \\ & 2x_1 + \;\; 4x_2 + \;\; 4x_3 & \geq & 3, & \quad x_i \geq 0, \; i = 1, 2, 3. \end{array}$$

Solve the dual problem using the simplex algorithm and hence, or otherwise, derive the optimal solution $\bar{x}_i$, $1 \leq i \leq 3$, of the above problem.

Suppose that the coefficients in the objective function of the primal problem change from $(7, 10, 11)$ to $(7 + \epsilon_1, 10 + \epsilon_2, 11 + \epsilon_3)$. Give necessary and sufficient conditions on ϵ_1, ϵ_2 and ϵ_3 which ensure that the optimal solution remains $\bar{x}_i$, $1 \leq i \leq 3$.

95405

(a) Shops A,B and C require 8, 10 and 6 units of some goods respectively and these are to be supplied by warehouses D, E and F with supplies of 7, 7 and 10 respectively. The costs per unit of shipping goods between shops and warehouses are given in the table below. Use the transportation algorithm to find the minimal shipping cost.

	D	E	F
A	5	3	6
B	7	1	8
C	9	4	7

(b) State and prove the Lagrangian sufficiency theorem.
Illustrate the use of this theorem by solving the following problem.

$$\text{Minimize } 2x_1 + 2x_2 + x_3^2 \text{ subject to } x_1^2 + x_2^2 + 2x_3 = 4.$$

If the right-hand side of the constraint is altered to $4+\epsilon$, what is the change in the optimal value of the objective function for small values of ϵ?

94305

(a) Determine the value and optimal strategies for the 2-person zero-sum game with payoff matrix

$$\begin{pmatrix} 2 & 3 \\ 5 & a \end{pmatrix}$$

for each value of a, $-\infty < a < \infty$. Here the (i, j) entry of the matrix is the payoff to the row player if row i and column j are chosen.

(b) Consider the problem:

$$\begin{aligned} \text{Minimize } & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to } & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,\dots,m, \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,\dots,n, \\ & x_{ij} \geq 0, \quad \text{for all } i,j, \end{aligned}$$

where $a_i \geq 0$, $b_j \geq 0$ satisfy $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Formulate the dual of this problem and state necessary and sufficient conditions for optimality. Explain briefly how these conditions lead to a simple algorithm for solving such problems.

Four power stations may supply electricity at rates 8, 14, 15, 8, respectively. There are three consumption points whose requirements are 13, 21, and 7, respectively. Routing costs per unit amount per unit time are shown in the table below. Determine a minimal cost routing scheme from sources to consumption points and the minimal cost.

		Sources			
		1	2	3	4
Consumption	1	4	3	2	2
points	2	3	5	8	6
	3	11	3	9	4

94405

(a) State and prove the Max-flow Min-cut Theorem.

(b) Use the two-phase simplex method to solve the problem:

$$\begin{array}{llcccccl} \text{Maximize} & 3x_1 & + & 4x_2 & + & x_3 & & \\ \text{subject to} & 2x_1 & + & x_2 & & & \leq & 5 \\ & 3x_1 & + & 2x_2 & + & x_3 & = & 6 \\ & x_1 & + & 4x_2 & + & x_3 & \leq & 7 \\ & \multicolumn{7}{l}{x_i \geq 0, \quad i = 1,2,3.} \end{array}$$

Write down the dual linear programming problem and the complementary slackness conditions. Hence, or otherwise, find the optimum solution to the dual problem.

93305

(a) By using Lagrange's sufficiency theorem, find points in $\mathbb{R}^3$ at which the plane parallel to $x_1 + 2x_2 + x_3 = 0$ is tangent to the ellipsoid

$$\frac{x_1^2}{3} - \frac{x_1 x_2}{2} + \frac{x_2^2}{4} + x_3^2 = 1.$$

(b) For a linear programming problem

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{x} \geq 0,\ A\mathbf{x} = \mathbf{b}$$

define the *feasible set* and its *extreme points*. Prove that the feasible set is convex. State and prove the relation between optimal solutions and extreme points of the feasible set.

Consider the following simplex tableau for a linear programming problem (1), with $x = (x_1, \dots, x_6)$:

x_1	x_2	x_3	x_4	x_5	x_6	f
4	2	-1	1	0	0	3
1	3	-2	0	1	0	1
5	1	β	0	0	1	2
-1	-1	α	0	0	0	-6

What is the next step in the simplex algorithm in each of the following cases:

(i) $\alpha = -1$, $\beta = 1$,

(ii) $\alpha = 1$, $\beta = -1$,

(iii) $\alpha = 1$, $\beta = 1$?

Justify your answers.

93405

(a) By using the transportation algorithm, or otherwise, solve the following linear programming problem.

$$\begin{aligned}
&\text{Minimize } 7x_1 + 4x_2 + 9x_3 + 8x_4 + 12x_5 + 5x_6 + 3x_7 + 11x_8 + 7x_9\\
&\text{subject to } x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0,\ x_8 \geq 0,\ x_9 \geq 0\\
&\text{and } x_1 + x_2 + x_3 = 17,\ x_4 + x_5 + x_6 = 39,\ x_7 + x_8 + x_9 = 15,\\
&x_1 + x_4 + x_7 = 36,\ x_2 + x_5 + x_8 = 31,\ x_3 + x_6 + x_9 = 4.
\end{aligned}$$

(b) Show that the problem of finding an optimal strategy $p = (p_1, \dots, p_n)$ for a two-person zero-sum game with payoff matrix $A = (a_{ij}, 1 \leq i, j \leq n)$ can be written as

$$\begin{aligned}
&\text{maximize } v\\
&\text{subject to } v \leq \sum_i a_{ij} x_i,\ 1 \leq j \leq n,\ v \in \mathbb{R}^1,\ x_1 \geq 0, \dots, x_n \geq 0 \text{ and } \sum_i x_i = 1.
\end{aligned}$$

Determine the dual of this problem and give the interpretation of the dual optimal solution q. Prove that if v_1 is the optimal value in the primal problem and v_2 is the optimal value in the dual problem then $v_1 \leq v_2$. State and prove a sufficient condition on p, q and $w = p^T A q$ for optimality. Give the interpretation of w.

[*General results may be used without proofs, but must be carefully formulated.*]

92305 sample

(a) Define an extreme point of a convex set $X \subset \mathbb{R}^n$. Define a basic feasible solution for the linear programming problem

$$\max\ \mathbf{c}^T\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

State a theorem giving the connection between basic feasible solutions and extreme points of the feasible set for the problem above. Explain, in a few sentences, why basic feasible solutions are important in solving linear programming problems. [*You may assume that there is no in the problems under consideration.*]

(b) Solve the linear programming problem

$$\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & 6x_2 & - & x_3 & & \\
\text{subject to} & x_i \geq 0 & & & & & & \text{and} \\
& -x_1 & + & 4x_2 & - & x_3 & \leq & 2, \\
& 2x_1 & + & x_2 & + & x_3 & \leq & 5, \\
& -x_1 & + & x_2 & + & 2x_3 & \leq & 1.
\end{array}$$

Write down the dual problem and its solution.

92405 sample

(a) State the Lagrangian Sufficiency theorem and use it to derive a set of sufficient conditions for a flow $\mathbf{x} = (x_{ij})$ to be a minimal cost feasible circulation in a network. [*Consider, as usual, a network with set of nodes* $N = \{1, 2, \ldots, n\}$ *and the set of directed arcs* $A = \{(i,j); 1 \leq i, j \leq n, i \neq j\}$. *Suppose that on each arc* (i,j) *there is a flow* x_{ij} *satisfying an inequality of the form* $m_{ij} \leq x_{ij} \leq M_{ij}$ *and that the cost per unit flow in arc* (i,j) *is* c_{ij}.]

(b) Sources 1, 2, 3 stock chocolate in amounts of 20, 42, 18 tons respectively. The demands for chocolate at destination 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

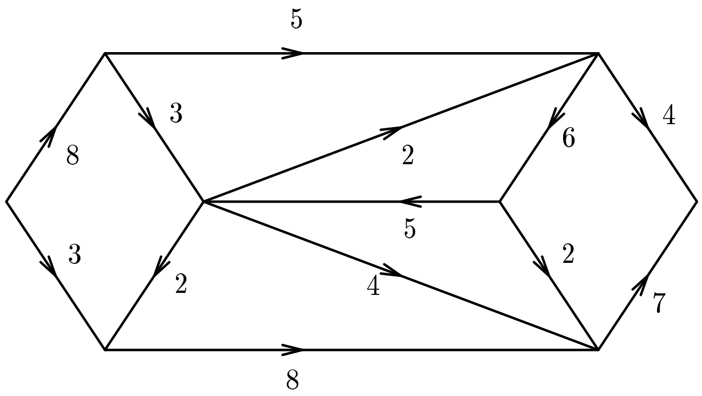
$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (i,j) entry corresponding to the route $i \to j$.

Describe the transportation algorithm. Illustrate your answer by using the algorithm to find the optimal transportation scheme and the associated minimal cost for the problem stated above.

92107 For a network with capacitated links, source a and sink b, define the values of the source arrival and sink departure fluxes. Define the net flux in a flow and the capacity of a cut and prove the relation between these. Prove the relation between the maximal net flux and the minimum value of a cut.

Find a maximal flow and minimal cut for the example below.



92207

Minimize

$$f(x_1, x_2) = 4x_1^2 + 2\sqrt{3}x_1x_2 + 2x_2^2$$

subject to

$$\sqrt{21}x_1 + \sqrt{7}x_2 = 1$$

and

$$-x_1 + \sqrt{3}x_2 \geq -2.$$

92407

Solve the following linear programming problem: minimize

$$4x_1 + 2x_2 + x_3 + 3x_4,$$

for

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \tag{2}$$

subject to

$$x_1 + x_2 = 9, \quad x_1 + x_3 = 7, \quad x_3 + x_4 = 4.$$

Is it true that and solution to problem (1) gives a solution to the problem: minimize

$$4x_1 + 2x_2 + x_3 + 3x_4,$$

for

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \tag{3}$$

subject to

$$x_1 + x_2 \geq 9, \quad x_1 + x_3 \geq 7, \quad x_3 + x_4 \geq 4 \ ?$$

Justify your answer.

91107

Show how to reduce the problem of finding optimal strategies for a two-person zero-sum game with payoff matrix A to a pair of linear programming problems which are duals of one another.

Find optimal strategies for both players in the game with payoff matrix

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

91207

A gambler at a horse race has an amount b to bet. The gambler assesses p_i as the probability that horse i will win, and knows that s_i has been bet on horse i by the others, for $i = 1, 2, \ldots, I$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_I)$ to

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{I} \frac{p_i x_i}{s_i + x_i} \\ \text{subject to} & \displaystyle\sum_{i=1}^{I} x_i = b, \quad x_1, x_2, \ldots, x_I \geq 0, \end{array}$$

where x_i is the amount the gambler bets on horse i.

Find the form of the gambler's optimal strategy. Deduce that if b is small enough, the optimal strategy is to bet only on the horses for which the ratio p_i/s_i is maximal.

91407

What is meant by a transportation problem? Describe an algorithm for solving such a problem, illustrating your account by solving the problem with three sources and three destinations described by the table

	Destinations			
Sources	4	3	1	10
	6	10	3	8
	3	5	7	8
	3	9	14	

where the figures in the boxes denote transportation costs, the right-hand column denotes supplies, and the bottom row denotes requirements.

90106

Define the Lagrangian $L(x; \lambda)$ for the problem:

maximize $f(x)$ in $x \in X$ subject to $g(x) = b$, where f maps $\mathbb{R}^n$ to $\mathbb{R}$, g maps $\mathbb{R}^n$ to $\mathbb{R}^m$, and b, X are given, with $b \in \mathbb{R}^m$m $X \subset \mathbb{R}^n$.

Show that if there exist $\bar{x} \in X$ and $\bar{\lambda} \in \mathbb{R}^m$ such that

$$L(\bar{x}; \bar{\lambda}) \geq L(x; \bar{\lambda}) \quad \text{for all} \quad x \in X \quad \text{and} \quad g(\bar{x}) = b$$

then $\bar{x}$ is optimal for the above problem.

Use the above result to minimize $(v_1/x_1) + (v_2/x_2)$ subject to

$$x_1c_1 + x_2c_2 = c, \quad x_1 \geq a_1,\ x_2 \geq a_2,$$

where c_1, c_2 and a_1, a_2, v_1, v_2 and c are given positive constants, with $c \geq a_1c_1 + a_2c_2$.

90207

Consider the problem

$$\begin{array}{rrcl}
\text{minimize} & 5x_1 - 3x_2 & = & f \\
\text{in} & 2x_1 - x_2 + 3x_3 & \leq & 4 \\
 & x_1 + x_2 + 2x_3 & \leq & 5 \\
 & 2x_1 - x_2 + x_3 & \leq & 1 \\
 & x_1, x_2, x_3 & \geq & 0.
\end{array}$$

(a) Use the simplex method to solve the problem.

(b) Write down the dual problem, and solve it by inspection of the final tableau for the primal.

(c) If the constraints in the right hand side of the above problem are changed to $4 + \epsilon_1$, $5 + \epsilon_2$, $1 + \epsilon_3$, respectively, for small $\epsilon_1, \epsilon_2, \epsilon_3$, by how much does the optimum value of f change?

90407 Given the $m \times n$ matrix of costs (c_{jk}), write down the Lagrangian for the problem of choosing (x_{jk}) to minimize $\sum\sum c_{jk}x_{jk}$,

$$\begin{array}{rll}
\text{subject to } \displaystyle\sum_k x_{jk} = r_j & \text{for each } j \\
\displaystyle\sum_j x_{jk} = s_k & \text{for each } k \\
\text{and } x_{jk} \geq 0 & \text{for each } j \text{ and } k,
\end{array}$$

where (r_j), (s_k) are given, with $\sum r_j = \sum s_k$. Hence describe briefly the rationale of the transportation algorithm.

In the table below, the row totals correspond to the amounts of basic foodstuffs, in tons, available at each of 4 sources in the U.K. The column totals correspond to the amounts required, also in tons, at 3 Eastern bloc destination, and the corresponding costs are in the body of the table: note that an x in a cell indicates the corresponding source to destination route is prohibitively expensive. Use the transportation algorithm to find the cheapest way of transporting the required foodstuffs, describing your working clearly.

1	x	x	4
2	2	x	6
3	3	4	2
4	4	2	1
7	9	3	

89107

Let f be a convex function on a closed, convex set X of $\mathbb{R}^n$. Show that:

(a) Any local minimum on f on X is also a global minimum on X.
(b) The set on which f attains its global minimum is convex.
(c) The function f cannot have more than one strict local minimum.
(d) If f is non-constant, it cannot attain its global maximum at an interior point of X.

89207

Maximize $3x_1 + 6x_2 - x_3$ subject to $x_i \geq 0$ and

$$\begin{aligned} -x_1 + 4x_2 - x_3 &\leq 2, \\ 2x_1 + x_2 + x_3 &\leq 5, \\ -x_1 + x_2 + 2x_3 &\leq 1. \end{aligned}$$

Write down the dual problem and its solution.
Now replace the second constraint by

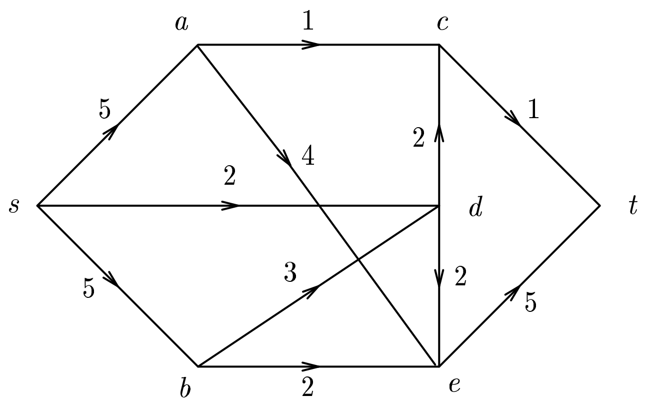
$$2x_1 + x_2 + x_3 \leq 5 + t,$$

where $t \geq 0$. Find the new maximum.

89407

For a capacitated network with source s and sink t define a value of a flow and the capacity of a cut. Show that the maximum value of a flow is the minimum value of a cut.

Find such a 'maximal flow' and 'minimal cut' for the network below.



88107

Maximize $2x_1 + x_2 + 3x_3 - 2x_4$ subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \end{aligned}$$

and $x_i \geq 0$ $(i = 1, \ldots, 4)$.

88207

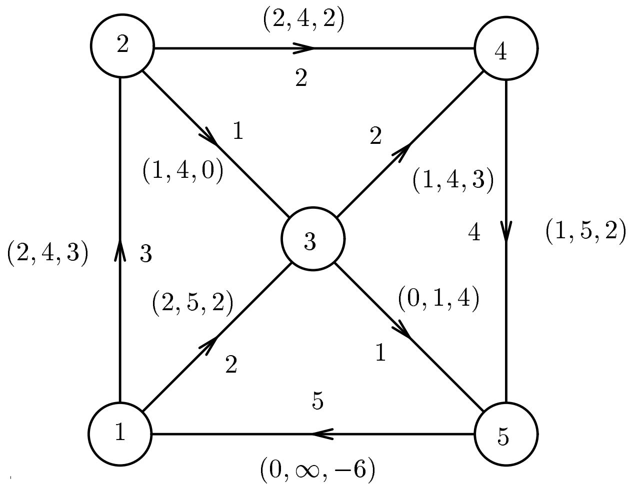
Show how to reduce the problem of finding optimal strategies for a game given by a matrix A to a pair of linear programming problems which are duals of one another.

The matrix of an $n \times n$ matrix game is such that the row and column sums all equal s. Show that the game has value s/n. [*You may assume that, if you can find* $\mathbf{x}, \mathbf{y}$ *and* v *such that* $\mathbf{y}^T A \geq v$, $A\mathbf{x} \leq v$ *and* $\mathbf{y}^T A\mathbf{x} = v$, *then* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal strategies and the value of the game is* v.]

88407

Consider a network with set of nodes $N = \{1, 2, \dots, n\}$ and set of directed arcs $A = \{(i,j) : 1 \leq i, j \leq n,\ i \neq j\}$. Suppose that on each arc (i,j) there is a flow x_{ij} satisfying an inequality of the form $a_{ij} \leq x_{ij} \leq b_{ij}$ and that the cost per unit flow in arc (i,j) is c_{ij}. Derive a set of sufficient conditions for a $x = (x_{ij})$ to be a minimal cost circulation in the network. [A flow $x = (x_{ij})$ is a *circulation* if $\sum_j x_{ij} = \sum_\ell x_{\ell i}$ for all i.]

By finding a suitable potential on the nodes of the network, or otherwise, show that the flow illustrated below is a minimal cost circulation. [Each arc is labelled with x_{ij} and with a triple of numbers giving the constants (a_{ij}, b_{ij}, c_{ij}) for that arc.]



87106

(a) Solve the problem:

$$\max x_1 + 2x_2 + 6x_3 + 2x_4,$$

subject to $x_i \geq 0$, and

$$\begin{array}{rrrrcl} 2x_1 & & +3x_3 & & \leq & 12 \\ x_1 + & x_2 & & + \ x_4 & \leq & 12 \\ x_1 + & 6x_2 & & - \ x_4 & \leq & 15 \\ & 3x_2 - & x_3 + & 2x_4 & \leq & 10. \end{array}$$

(b) Imagine that after you had completed the first part of the question, the examiner rushed breathlessly into the examination room and announced that the question contained a printing error, namely that the fourth constraint, $3x_2 - x_3 + 2x_4 \leq 10$, should not have been printed. *Without* starting the modified problem afresh, how would you change the final tableau you obtained for the original problem, so as to allow you to find the new optimum? Implement these changes and find the new optimum.

87306

Consider the transportation problem

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

subject to

$$\sum_j x_{ij} = s_i,$$
$$\sum_i x_{ij} = d_j,$$

where the index i runs through $\{1, \ldots, m\}$, the index j runs through $\{1, \ldots, n\}$, and the non-negative constants c_{ij}, s_i, d_j satisfy $\sum_i s_i = \sum_j d_j$. Describe briefly an algorithm for solving this problem.

The standard algorithm will not work properly if the problem is *degenerate*; that is, if there are proper non-empty subsets $A \subseteq \{1, \ldots, m\}$, $B \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in A} s_i = \sum_{j \in B} d_j.$$

Suppose that the constraints (s_i) and (d_j) are such that the problem is degenerate, but that for the constraints (s_i') and (d_j') the problem is not degenerate. Prove that there exists $\eta > 0$ such that for all $0 < \epsilon < \eta$, the problem is non-degenerate for the constraints $(s_i(\epsilon))$ and $(d_j(\epsilon))$ defined by

$$s_i(\epsilon) = s_i + \epsilon s_i',$$
$$d_j(\epsilon) = d_j + \epsilon d_j'.$$

For each $0 < \epsilon < \eta$, let $x_{ij}(\epsilon)$ be an optimal solution to the transportation problem with constraints $(s_i(\epsilon))$ and $(d_j(\epsilon))$. Deduce that there exists a sequence (ϵ_n) decreasing to zero such that

$$\lim_{n \to \infty} x_{ij}(\epsilon_n) = \bar{x}_{ij}$$

exists, and is an optimal solution to the original (degenerate) problem with constraints (s_i) and (d_j).

87406

What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is convex? Prove that a convex function is continuous, and that for each $a \in \mathbb{R}$ there exists $\gamma \in \mathbb{R}$ such that

$$f(t + a) \geq f(a) + \gamma t$$

for all $t \in \mathbb{R}$. Is γ necessarily unique?

Now suppose that f is convex and that X is a bounded random variable. Prove the inequality

$$E(f(X)) \geq f(E(X)).$$

86106

(a) The controller of a chemical plant has to decide the level $x_i \geq 0$ of production of each of n chemicals. Unit production of the ith chemical consumes an amount a_{ji} of the jth resource ($j = 1, \dots, d$) and results in profit c_i. The plant has stock b_j of the jth resource. Supposing that there is an additional setting up cost $\gamma_i > 0$ to be paid if and only if the level x_i is strictly positive, show how the problem of choosing the x_i so as to maximize the profit $\sum_{i=1}^{n} c_i x_i - \sum_{\{i: x_i > 0\}} \gamma_i$ can be reduced to a search through a finite set.

(b) Solve the following linear programme:

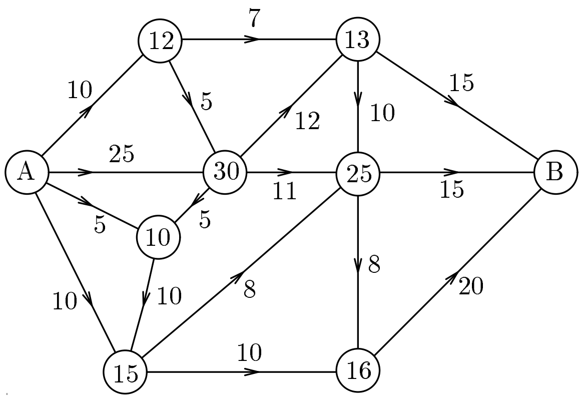
Maximize $8x_1 - 2x_2 + 10x_3$ subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 4, \\ -x_1 + 3x_3 &\leq 8, \\ x_2 - x_3 &\leq 4 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$.

86306

The road network between two towns A and B is represented in the diagram below. Each road is marked with an arrow giving the direction of the Each road is marked with an arrow giving the direction of flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity of the village.

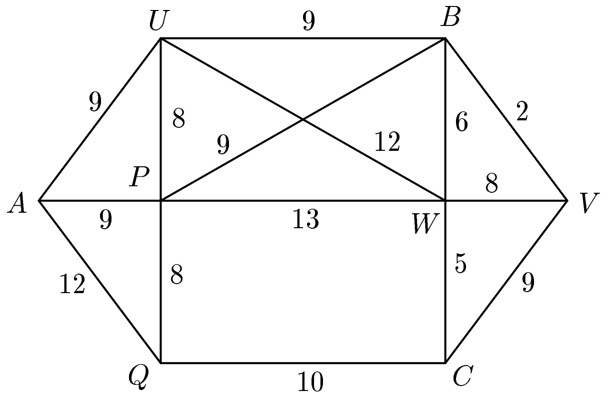


Explaining briefly the method you use, obtain the maximal flow from A to B, and draw your maximal flow.

The Minister of Transport intends to build a by-pass round one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if her intent is to increase the maximal flow from A to B as much as possible? What will the maximal flow be once the by-pass has been built?

85106

A political organisation has 24 members living in three towns A, B and C. Town A has 10 members, town B has 5, and town C has 9. On a particular day, rallies are to be held in towns U, V and W, and the organisation wishes to send 8 members to take part in each rally. The diagram below shows schematically the rail links between these towns and towns P and Q. The number beside each link is the rail fare per person (in either direction) between the two towns at the end of the link. Find a least-cost allocation and routing of the members to rallies, and determine the corresponding cost. Give brief description of the procedure you adopt for solving this problem.



85306

Consider the following constrained minimization problem and call it the *primal* problem:

Minimize	$f(x) = c_1x_1 + \cdots + c_nx_n$
subject to	$a_{i1}x_1 + \cdots + a_{in} \leq b_i \quad (i = 1, \ldots, m)$
and	$x_j \geq 0 \quad (j = 1, \ldots, n)$.

Write down the *dual* problem. Define what is meant by *feasible solution* and sow that if there is a feasible solution to each problem then both problems have finite optimal solutions and the values of the objective functions for these solutions are equal.

85406

Let X be a subset of $\mathbb{R}^n$ and b a point in $\mathbb{R}^m$, where $m < n$. Consider the problem of minimizing the function $f(x)$ for $x \in X$ subject to the constraints $g(x) = b$, where f maps $\mathbb{R}^n$ into $\mathbb{R}$ and g maps $\mathbb{R}^n$ into $\mathbb{R}^m$. Define the Lagrangian $L(x; \lambda)$ for this problem, and show that it there exists $x^* \in X$ and $\lambda^* \in \mathbb{R}^m$ such that

$$L(x^*; \lambda^*) \leq L(x; \lambda^*) \quad \text{for all } x \in X$$

and

$$g(x^*) = b$$

then x^* is optimal.

Two sets of n positive real numbers $(p_1, \ldots, p_n)$, $(q_1, \ldots, q_n)$ are given, where the numbers in each set sum to 1 (i.e., they are probability distributions). It is desired to find numbers $x_1, \ldots, x_n$, where $0 \leq x_i \leq 1$, such that for a given number α, $(0 < \alpha < 1)$ the quantity

$$\beta = 1 - (x_1 q_1 + \cdots + x_n q_n)$$

is minimized, subject to

$$x_1 p_1 + \cdots + x_n p_n = \alpha\,.$$

Write down the Lagrangian for this problem, and show that the minimization is achieved if, for a certain value of λ,

$$\begin{array}{lll} x_i = 1 & \text{if} & q_i > \lambda p_i \\ x_i = 0 & \text{if} & q_i < \lambda p_i \\ x_i = \phi & \text{if} & q_i = \lambda p_i \end{array}$$

for a value of ϕ such that $0 < \phi \leq 1$.

[**Hint**. Without loss of generality, suppose that $q_1/p_1, \ldots, q_n/p_n$ are in increasing order of magnitude.]