D2: Optimization

Examples Sheet

It is recommended that for your first supervision you attempt questions for Lectures 1–6 and that for your second supervision you attempt questions for Lectures 7–12. You should later do questions F1–F18 to gain further practice and to revise. A copy of this sheet can be found from the WWW site: `http://www.statslab.cam.ac.uk/~rrw1/opt/index.html`.

Linear programming: introduction (Lectures 1–2)

Questions in this section are introductory and cover, for a different problem, the same material as the first two lectures of the course. If you attended lectures (and understood them) you may wish to omit some or all of these questions.

1. Let P be the linear problem

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & - & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0. \end{array}$$

Solve P graphically in the x_1, x_2 plane.

2. Write P in matrix notation: max $c^T x$ subject to $Ax \leq b$, $x \geq 0$ and write down the problem D defined by: min $b^T \lambda$ subject to $A^T \lambda \geq c$, $\lambda \geq 0$.

3. Introduce slack variables x_3, x_4, x_5 and extend c and A to rewrite P as

$$\max\ c_e^T x_e \quad \text{subject to } A_e x_e = b,\ x_e \geq 0.$$

4. Define a basic solution to a set of linear equations. How many basic solutions are there to the equation $A_e x_e = b$? Define a basic feasible solution. How many of these are there? Are all the basic solutions non-degenerate?

5. Write down the values of the variables $x_1, \ldots, x_5$ and of the objective function for all basic solutions to the equation $A_e x = b$ of question 3. Mark these on your diagram. Which are the basic feasible solutions?

6. Introduce slack variables λ_4 and λ_5 into the problem D from question 2. Write down the value of the variables $\lambda_1, \ldots \lambda_5$ and of the objective function for each basic solution of D. Which are the basic feasible solutions of D?

7. Let $z_1 = x_3$, $z_2 = x_4$, $z_3 = x_5$ be the slack variables for P and $v_1 = \lambda_4$, $v_2 = \lambda_5$ be the slack variables for D. Show that for each basic solution x of P there is exactly one basic

solution λ of D such that (i) $c_e^T x_e = b^T \lambda$ (same objective value) and (ii) $\lambda_i z_i = 0$, $i = 1, 2, 3$ and $x_j v_j = 0$, $j = 1, 2$ (complementary slackness).

For how many pairs $\{x_e, \lambda\}$ is x_e feasible for P and λ feasible for D?

8. Solve problem P using the simplex algorithm starting with initial basic feasible solution $x_1 = x_2 = 0$. Try both choices of the variable to put into the basis on the first step. Compare the objective rows of the various tableaux generated with appropriate basic solutions to problem D? What do you observe?

Linear Programming: simplex, two-phase algorithm (Lectures 3–4)

LP1. Use the simplex algorithm to solve

$$\begin{array}{lrcl} \text{maximize} & 3x_1 + x_2 + 3x_3 & & \\ \text{subject to} & 2x_1 + x_2 + x_3 & \leq & 2 \\ & x_1 + 2x_2 + 3x_3 & \leq & 5 \\ & 2x_1 + 2x_2 + x_3 & \leq & 6 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

Each row of the final tableau is the sum of scalar multiples of the rows of the initial tableau. Explain how to determine the scalar multipliers directly from the final tableau.

LP2. Define P(ϵ) to be the LP problem obtained by replacing the vector $b = (4, 4, 1)^T$ in P (question 1) by the perturbed vector $b(\epsilon) = (4 + \epsilon_1, 4 + \epsilon_2, 1 + \epsilon_3)^T$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, for the optimal value for P(ϵ) when the ϵ_i are small. If $\epsilon_2 = \epsilon_3 = 0$, for what range of ϵ_1 values does the formula hold?

LP3. Apply the simplex algorithm to

$$\begin{array}{lrcl} \text{maximize} & x_1 + 3x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \leq & 4 \\ & -x_1 + x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Explain what happens with the use of a diagram.

LP4. Use the two-phase algorithm to solve:

$$\begin{array}{lrcl} \text{maximize} & -2x_1 - 2x_2 & & \\ \text{subject to} & 2x_1 - 2x_2 & \leq & 1 \\ & 5x_1 + 3x_2 & \geq & 3 \\ & x_1, x_2 & \geq & 0. \end{array}$$

[Hint: You should get $x_1 = \frac{9}{16}$, $x_2 = \frac{1}{16}$. Note that it is possible to choose the first pivot column so that phase I last only one step. But this requires a different choice of pivot column than the one specified by the usual rule-of-thumb.]

LP5. Use the two-phase algorithm to solve:

$$\begin{array}{lrcl} \text{minimize} & 13x_1 + 5x_2 - 12x_3 & & \\ \text{subject to} & 2x_1 + \ x_2 + \ 2x_3 & \leq & 5 \\ & 3x_1 + 3x_2 + \ \ x_3 & \geq & 7 \\ & x_1 + 5x_2 + \ 4x_3 & = & 10 \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

Lagrangian methods and duality (Lectures 5–6)

L1. Minimize each of the following functions in the region specified.

(a) $3x$ in $\{x : x \geq 0\}$; (b) $x^2 - 2x + 3$ in $\{x : x \geq 0\}$; (c) $x^2 + 2x + 3$ in $\{x : x \geq 0\}$.

For each of the following functions specify the set Y of λ values for which the function has a finite minimum in the region specified, and for each $\lambda \in Y$ find the minimum value and (all) optimal x.

(d) λx subject to $x \geq 0$; (e) λx subject to $x \in \mathbb{R}$; (f) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \in \mathbb{R}$;
(g) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \geq 0$; (h) $(\lambda_1 - \lambda_2)x$ subject to $0 \leq x \leq M$.

L2. Use the Lagrangian Sufficiency Theorem to minimize ${x_1}^2 + 2{x_2}^2$ subject to $x_1 + 3x_2 = b$.

L3. (a) Use the Lagrangian Sufficiency Theorem to solve:

$$\begin{array}{lrcl} \text{maximize} & x + 2y + z & & \\ \text{subject to} & x^2 + y^2 + z^2 & \leq & 1 \\ & x, y, z & \geq & 0. \end{array}$$

(b) Find a point in $\mathbb{R}^3$ at which a plane parallel to $x + 2y + z = 0$ is tangent to the unit sphere.

L4. Minimize $f = \sum v_i x_i^{-1}$ in $x \geq 0$ subject to $\sum a_i x_i \leq b$ where $a_i, v_i > 0$ for all i and $b > 0$. [In this example f is the variance of an estimate derived from a stratified sample survey subject to a cost constraint; x_i is the size of the sample for the i^{th} stratum, the a_i and v_i are measures of sampling cost and of variability for this stratum.] Check that the change in the minimal variance f for a small change δb in available resources is $\lambda\ \delta b$ where λ is the Lagrange multiplier.

Linear programming and duality (Lecture 6)

D1. Write down the Lagrangian for each of the following problems. In each case find the set Y of λ values for which the Lagrangian has a finite minimum (subject to the appropriate regional constraint), calculate the minimum $L(\lambda)$ for each $\lambda \in Y$, and write down the dual problem. In each case, write down the conditions for primal and dual feasibility and any additional conditions (the complementary slackness conditions) needed for optimality.

(a) min $c^T x$ subject to $Ax \leq b$, $x \geq 0$; (b) min $c^T x$ subject to $Ax = b$, $x \geq 0$.

D2. For each of the problems in the previous question verify that the dual of the dual problem is the primal problem.

Suppose that an LP problem P is written in the two equivalent forms

$$\min\ c^T x \quad \text{subject to} \quad Ax \leq b, \quad x \geq 0,$$

where A is an $m \times n$ matrix, $c, x \in \mathbb{R}^n$ and

$$\min\ c_e^T x_e \quad \text{subject to} \quad A_e x_e = b, \quad x_e \geq 0,$$

where, after addition of slack variables and extension of the matrix A and vector c in the appropriate way A_e is $m \times (n+m)$, and $c_e, x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of problem P, and show that the dual problems are equivalent to each other.

D3. Consider the problem

$$\begin{array}{lrcl} \text{minimize} & 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 & & \\ \text{subject to} & x_1 + \ x_2 + 2x_3 + \ x_4 + 3x_5 & \geq & 4 \\ & 2x_1 - 2x_2 + 3x_3 + \ x_4 + \ x_5 & \geq & 3 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 & \geq & 0. \end{array}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.

Algebra of linear programming (Lectures 7–8)

LP6. Consider the three equations in 6 unknowns given by $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Choose B $= \{1, 3, 6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_3, x_6)^T$, $x_N = (x_2, x_4, x_5)^T$ and the matrices A_B and A_N are constructed appropriately.

Now, write $c^T x = c_B^T x_B + c_N^T x_N$ and hence write $c^T x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate the variables x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. For $c = (3, 1, 3, 0, 0, 0)^T$ write $c^T x$ in terms of the non-basic variables. Prove directly from the formula for $c^T x$ that the basic solution you have computed is optimal for the problem maximize $c^T x$ subject to $x \geq 0$, $Ax = b$.

Compare your answer with your answer to question LP1 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$. Why is it not fair to say that the simplex algorithm is just a complicated way to invert a matrix?

LP7. (Degeneracy.) Take problem P from question 1 and add the additional constraint $x_1 + 3x_2 \le 6$. Use the simplex algorithm putting x_2 into the basis at the first stage. Try each of the possibilities for the variable leaving the basis. Explain, with a diagram, what happens.

LP8. In the previous question the additional constraint was redundant (it did not alter the feasible set). Can degeneracy occur without redundant equations?

LP9. Show that introducing slack variables in a LP does not change the extreme points of the feasible set by proving (using the definition of an extreme point) that x is an extreme point of $\{x : x \ge 0, Ax \le b\}$ if and only if $\binom{x}{z}$ is an extreme point of $\{\binom{x}{z} : x \ge 0, z \ge 0, Ax + z = b\}$.

Game theory (Lecture 9)

G1. Give sufficient conditions for strategies $\mathbf{p}$ and $\mathbf{q}$ to be optimal for a two-person zero-sum game with pay-off matrix A and value v.

Two players fight a duel: they face each other $2n+1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends. Otherwise both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the i^{th} pace forward $(i = 0, 1, \dots, n)$ is i/n. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired.

Show that, if $n = 4$, the strategy 'shoot after taking two steps' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11})$.]

G2. Optimal strategies are not always unique as illustrated by the following payoff matrix,

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}.$$

Find all optimal strategies.

G3. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n. [Hint: Guess a solution and show that it is optimal.]

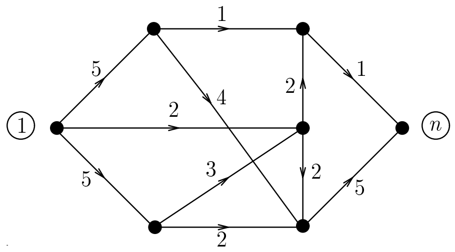
G4. Find optimal strategies for both players, and the value of the game, for the game with payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

[You may like to try to compare the effort required to solve this by (a) seeking strategies and a value which satisfy the optimality conditions (b) direct solution of Player I's original minimax problem and (c) using the simplex method on one of the players problems after transforming it as suggested in the lectures.]

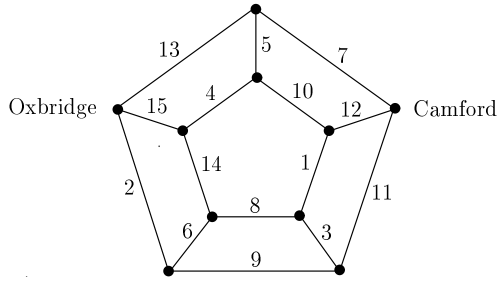
Max-flow/min-cut problems (Lecture 10)

N1. Show that the maximal value of a flow through a network with a source at node 1 and a sink at node n equals the minimum cut capacity. Find a 'maximal flow' and 'minimal cut' for the network below.



N2. Devise rules for a version of the Ford-Fulkerson algorithm which works with undirected arcs.

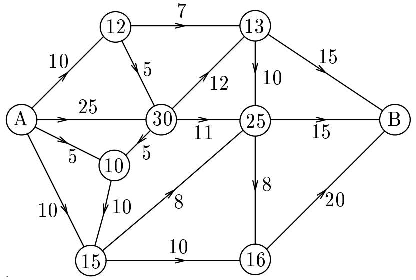
As a consequence of drought, an emergency water supply must be pumped from Oxbridge to Camford along the network of pipes shown in the figure. The numbers against the pipes show their maximal capacities, and each pipe may be used in either direction. Find the maximal flow and prove that it is maximal.



N3. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal flow problems?

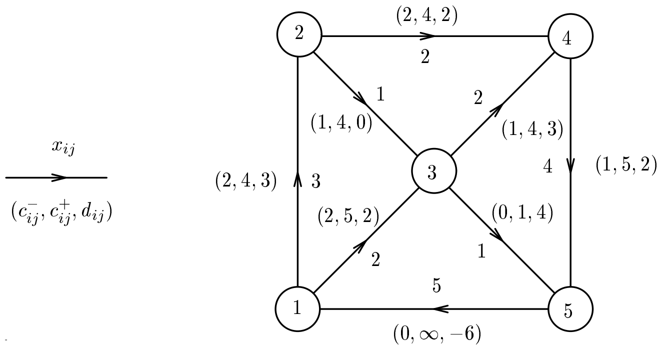
The road network between two towns A and B is represented in the diagram below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.

The Minister of Transport intends to build a by-pass round one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if her intent is to increase the maximal flow from A to B as much as possible? What would the new maximal flow be?



Minimal cost circulations (Lecture 11)

N4. By finding a suitable potential on the nodes (i.e., a set of suitable node numbers) of the network show that the flow illustrated below is a minimal cost circulation. [Each arc is labelled with x_{ij} and with a triple of numbers giving the constants $(c_{ij}^-, c_{ij}^+, d_{ij})$ for that arc.]



N5. Consider the problem of assigning lecturers $L_1, \dots, L_n$ to courses $C_1, \dots, C_n$ so as to minimize student dissatisfaction. The dissatisfaction felt by students if lecturer L_i gives course C_j is d_{ij}, and each lecturer must give exactly one course. Show how to state this problem as the problem of minimizing the cost of a circulation in a network. (Can you be sure that your network problem has an optimal flow of the appropriate kind?)

For the example of 3 lecturers and 3 courses with dissatisfaction matrix

$$\begin{pmatrix} 6 & 3 & 1 \\ 5 & 4 & 3 \\ 8 & 3 & 2 \end{pmatrix}$$

find an optimal flow through the appropriate network (by guessing) and compute node numbers for each node so that the optimality conditions are satisfied.

Transportation algorithm (Lecture 12)

N6. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 18 tons respectively. The demands for candy-floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (ij) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the associated minimal cost.

Further Examples

Linear programming, simplex algorithm (Lectures 3–4)

F1. Consider the problem $\min \sum_{i=0}^n |x_i|$ subject to $Ax \le b$ where A is $m \times n$ and $b \in \mathbb{R}^m$. Show how to convert the problem into a standard LP problem suitable for solution by the simplex algorithm. What happens if you replace min by max?

F2. Maximize $3x_1 + 6x_2 - x_3$ subject to $x_i \ge 0$ and

$$\begin{aligned} -x_1 + 4x_2 - x_3 &\le 2 \\ 2x_1 + x_2 + x_3 &\le 5 \\ -x_1 + x_2 + 2x_3 &\le 1. \end{aligned}$$

Write down the dual problem and its solution. Now replace the second constraint by $2x_1 + x_2 + x_3 \le 5 + t$. Find the new maximum for small values of t. For what range of t values is this solution valid?

F3. Use the simplex algorithm to solve

$$\begin{aligned} \text{minimize} \quad 5x_1 - 3x_2 &= f \\ 2x_1 - x_2 + 3x_3 &\le 4 \\ x_1 + x_2 + 2x_3 &\le 5 \\ 2x_1 - x_2 + x_3 &\le 1 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

Write down the dual problem, and solve it by inspection of the final tableau for the primal. If the constraints in the right hand side of the above problem are changed to $4+\varepsilon_1$, $5+\varepsilon_2$, $1+\varepsilon_3$, respectively, for small $\varepsilon_1, \varepsilon_2, \varepsilon_3$, by how much does the optimum value of f change?

Lagrangian sufficiency theorem (Lectures 5–8)

F4. Define the Lagrangian $L(x;\lambda)$ for the problem: minimize $f(x)$ in $x \in X$ subject to $g(x) = b$, where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $b \in \mathbb{R}^m$, $X \subset \mathbb{R}^n$.

Show that if there exist $\overline{x} \in X$ and $\overline{\lambda} \in \mathbb{R}^m$ such that

$$L(\overline{x};\overline{\lambda}) \le L(x;\overline{\lambda}) \quad \text{for all} \quad x \in X$$

and $g(\overline{x}) = b$ then $\overline{x}$ is optimal for the above problem.

Use the above result to minimize $(v_1/x_1) + (v_2/x_2)$ subject to $x_1c_1 + x_2c_2 = c$, $x_1 \ge a_1$, $x_2 \ge a_2$, where c_1, c_2 and a_1, a_2, v_1, v_2 and c are given positive constants, and $c \ge a_1c_1 + a_2c_2$.

F5. A gambler at a horse race has an amount b to bet. The gambler assesses p_i, the probability that horse i will win, and knows that s_i has been bet on horse i by others, for $i = 1, 2, \dots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gamblers optimal strategy is to choose $(x_1, x_2, \dots, x_n)$ to

$$\max \sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = b, \quad x_1, x_2, \dots, x_n \ge 0\,,$$

where x_i is the amount the gambler bets on horse i.

Find the form of the gamblers optimal strategy. Deduce that if b is small enough, the optimal strategy is to bet only on the horses for which the ratio p_i/s_i is maximal.

F6. Two sets of n positive real numbers $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ are given, where the numbers in each set sum to 1 (i.e., they are probability distributions). It is desired to find numbers $x_1, \dots, x_n$, where $0 \le x_i \le 1$, such that for a given number α, $(0 < \alpha < 1)$ the quantity

$$\beta = 1 - (x_1q_1 + \dots + x_nq_n)$$

is minimized, subject to

$$x_1p_1 + \dots + x_np_n = \alpha\,.$$

Write down the Lagrangian for this problem, and show that the minimization is achieved if, for a certain value of λ,

$$\begin{array}{lll} x_i = 1 & \text{if} & q_i > \lambda p_i \\ x_i = 0 & \text{if} & q_i < \lambda p_i \\ x_i = \phi & \text{if} & q_i = \lambda p_i \end{array}$$

for a value of ϕ such that $0 < \phi \le 1$.

[**Hint**. Without loss of generality, suppose that $q_1/p_1, \dots, q_n/p_n$ are in increasing order of magnitude. IB students should note that this is essentially a proof of the Neyman-Pearson lemma and should be able to give an interpretation to the numbers x_i.]

F7. Show how to solve the problem

$$\min \sum_{i=1}^{n} \frac{1}{(a_i + x_i)} \qquad \text{subject to} \qquad \sum_{i=1}^{n} x_i = b, \quad x_i \ge 0 \ \ (i = 1, \dots, n)$$

where $a_i > 0$, $i = 1, \dots, n$ and $b > 0$.

F8. A certain lecturer eats only nuts and beans. His pathetic daily salary is $I > 0$. The quality of his lectures is given by the real number $x_1^{a_1} x_2^{a_2}$ where x_1 is his daily consumption of nuts, x_2 is his daily consumption of beans, and the a_i are positive constants, $a_1 + a_2 = 1$. The price of nuts and beans are p_1 and p_2 respectively ($p_i > 0$). The lecturer, for reasons known best to himself, wishes to give lectures that are of as high a quality as possible, without spending more than his income. Unfortunately, the lecturer is too weak to solve his own optimization problem. Please do it for him.

Despite the lecturer's best efforts, the students still find his lectures pretty awful. They decide to make a collection and succeed in raising the tiny (even in comparison with I) amount δI. The sum having been handed over with due ceremony, how much of an increase in the quality of their lectures can the students expect?

F9. Consider the problem: minimize x^2 subject to $g(x) \ge b$. Show that it is possible to use the Lagrangian Sufficiency Theorem to solve the problem when (a) $g(x) = x$ and $b = 2$, but that this is not possible for (b) $g(x) = x^3$ and $b = 8$. Explain why the Lagrangian method works in case (a) but not in case (b).

Two-person zero-sum games (Lecture 9)

F10. Follow the usual steps to find the dual problem to player I's problem

$$\max\ v \qquad \text{subject to} \qquad \sum_i a_{ij} p_i \ge v \quad (\text{each } j), \quad \sum_i p_i = 1, \quad p \ge 0.$$

Show that the dual problem is player II's problem, and use the Lagrange sufficiency theorem to find sufficient conditions for a strategy vectors $\mathbf{p}$ and $\mathbf{q}$ to be optimal for the two players.

F11. Let the matrix (a_{ij}), $i = 1, \dots, m$, $j = 1, \dots, n$ define a two-person zero-sum game. Show that there exist numbers $p_1, \dots, p_m$ and $q_1, \dots, q_n$ with

$$p_i \ge 0, \qquad q_j \ge 0, \qquad \sum p_i = 1, \qquad \sum q_j = 1$$

such that

$$\sum_{i,j} a_{ij} p'_i q_j \le \sum_{i,j} a_{ij} p_i q_j \le \sum_{i,j} a_{ij} p_i q'_j$$

for all numbers $p'_1, \dots, p'_m$ and $q'_1, \dots, q'_n$ with

$$p'_i \ge 0, \qquad q'_j \ge 0, \qquad \sum p'_i = 1, \qquad \sum q'_j = 1$$

and explain the sense in which this entails the existence of optimal strategies for the game.

In the game of Undercut each player selects secretly a number from $1, 2, \dots, N$. The numbers are then revealed, and the player with the smaller number wins £1, unless the numbers are either *adjacent*, when the player with the larger number wins £2, or *equal*, when the game is tied (with payoff zero). Show that for $N \ge 5$ it is optimal for both players to name the numbers $1, 2, 3, 4, 5$ with probabilities $(\frac{1}{16}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{16})$ respectively, and to avoid all larger numbers.

Show, using the first part of the question or otherwise, that the optimal strategy given above is the *unique* optimal strategy for this game.

Maximal flows (Lecture 10)

F12. A network with nodes $1, \dots, n$ has a maximal capacity $M_{ij} \ge 0$ for transfer along the directed arc from node i to node j. Describe an algorithm to compute the maximal flow from node 1 to node n in the case when all the M_{ij} are integers. Prove that the algorithm will converge to the maximal flow in a finite number of steps.

Derive the Max-flow-Min-cut Theorem, and explain how the theorem can be used to prove that a particular flow is maximal.

F13. Formulate the maximum flow through a network problem as a linear programming problem. How many variables and constraints may be needed for a problem with n nodes? Show that the dual problem has a solution in which the variables takes only two values, and explain the significance of this result.

Applications of max-flow min-cut theorem (Lecture 10)

F14. (Konig-Egervary Theorem). Consider an $m \times n$ matrix A in which every entry is either 0 or 1. Say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set. Say that a set of 1's is *independent* if no pair of 1's of the set lies in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1's equals the minimal number of lines that cover the matrix.

F15. Derive the vertex form of Menger's Theorem, which states that if A and B are nodes of an undirected network then the maximum number of node-disjoint paths from A to

B which can be chosen simultaneously is equal to the minimum number of nodes whose removal disconnects A and B. [Two paths from A to B are node-disjoint if the only nodes they have in common are A and B. The removal of a set of nodes S disconnects A and B if any path from A to B passes through at least one node of S.]

F16. Suppose that N is a network with vertices $0, 1, 2, \dots, 2n, 2n+1$, where 0 is the source and $2n+1$ is the sink, such that

1. (a) for each $i = 1, \dots, n$, $(0, i)$ is an edge of capacity 1;
2. (b) for each $j = n+1, \dots, 2n$, $(j, 2n+1)$ is an edge of capacity 1;
3. (c) the only other edges have capacity n and are of the form (i, j) with $i \in \{1, \dots, n\}$, $j \in \{n+1, \dots, 2n\}$,

and for each subset $I \subset \{1, \dots, n\}$ the number of distinct vertices j such that an edge (i, j) exists for some $i \in I$ is not less than $|I|$, the number of elements in I. Prove that any maximal flow in N has value n.

Hence show (the Hall 'Marriage Theorem') that if we have a set of n boys and a set of n girls, such that every subset B of the boys between them know at least $|B|$ of the girls, then they can pair off, each boy with a girl whom he knows.

Transportation algorithm (Lecture 12)

F17. What is meant by a transportation problem? Describe an algorithm for solving such a problem, illustrating your account by solving the problem with three sources and three destinations described by the table

4	3	1	10
6	10	3	8
3	5	7	8
3	9	14	

where the figures in the boxes denote transportation costs (per unit), the right-hand column denotes supplies, and the bottom row denotes demands.

F18. A manufacturer has to supply $\{5, 7, 9, 6\}$ units of a good in each of the next four months. He can produce up to 8 units each month on ordinary time at costs $\{1, 3, 4, 2\}$ per unit, and up to 3 extra each month on overtime at costs $\{2, 5, 7, 4\}$ per unit (where costs are given for each of the next four months). Storage costs are 1 per unit per month. He desires to schedule production to minimize costs over the four-month period. Formulate his problem as a transportation problem and solve it.

[Hint. Formulate this problem as one with 8 sources and 5 destinations.]