Paper 1, Section I

8H Optimization

Suppose that $Ax \leqslant b$ and $x \geqslant 0$ and $A^T y \geqslant c$ and $y \geqslant 0$ where x and c are n-dimensional column vectors, y and b are m-dimensional column vectors, and A is an $m \times n$ matrix. Here, the vector inequalities are interpreted component-wise.

(i) Show that $c^T x \leqslant b^T y$.

(ii) Find the maximum value of

$$6x_1 + 8x_2 + 3x_3 \quad \text{subject to} \quad \begin{aligned} &2x_1 + 4x_2 + x_3 \leqslant 10, \\ &3x_1 + 4x_2 + 3x_3 \leqslant 6, \\ &x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

You should state any results from the course used in your solution.

Paper 2, Section I

9H Optimization

Let $N = \{1, \ldots, n\}$ be the set of nodes of a network, where 1 is the source and n is the sink. Let c_{ij} denote the capacity of the arc from node i to node j.

(i) In the context of maximising the flow through this network, define the following terms: feasible flow, flow value, cut, cut capacity.

(ii) State and prove the max-flow min-cut theorem for network flows.

Paper 3, Section II
21H Optimization
(i) What does it mean to say a set $C \subseteq \mathbb{R}^n$ is convex?

(ii) What does it mean to say z is an extreme point of a convex set C?

Let A be an $m \times n$ matrix, where $n > m$. Let b be an $m \times 1$ vector, and let

$$C = \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$$

where the inequality is interpreted component-wise.

(iii) Show that C is convex.

(iv) Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at least $m+1$ indices i are such that $z_i > 0$. Show that z is not an extreme point of C. [*Hint: If $r > m$, then any set of r vectors in $\mathbb{R}^m$ is linearly dependent.*]

(v) Now suppose that every set of m columns of A is linearly independent. Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at most m indices i are such that $z_i > 0$. Show that z is an extreme point of C.

Paper 4, Section II
20H Optimization
A company must ship coal from four mines, labelled A, B, C, D, to supply three factories, labelled a, b, c. The per unit transport cost, the outputs of the mines, and the requirements of the factories are given below.

	A	B	C	D	
a	12	3	5	2	34
b	4	11	2	6	21
c	3	9	7	4	23
	20	32	15	11	

For instance, mine B can produce 32 units of coal, factory a requires 34 units of coal, and it costs 3 units of money to ship one unit of coal from B to a. What is the minimal cost of transporting coal from the mines to the factories?

Now suppose increased efficiency allows factory b to reduce its requirement to 20.8 units of coal, and as a consequence, mine B reduces its output to 31.8 units. By how much does the transport cost decrease?

Paper 1, Section I

8E Optimization

What is the maximal flow problem in a network?

Explain the Ford–Fulkerson algorithm. Why must this algorithm terminate if the initial flow is set to zero and all arc capacities are rational numbers?

Paper 2, Section I

9E Optimization

Consider the function ϕ defined by

$$\phi(b) = \inf\{x^2 + y^4 : \, x + 2\,y = b\}\,.$$

Use the Lagrangian sufficiency theorem to evaluate $\phi(3)$. Compute the derivative $\phi'(3)$.

Paper 3, Section II

21E Optimization

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game. What is Player I's optimization problem?

Write down a sufficient condition that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy of Player II and the value of the game. If $m = n$ and A is an invertible, symmetric matrix such that $A^{-1}e \geqslant 0$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^m$, show that the value of the game is $(e^T A^{-1} e)^{-1}$.

Consider the following game: Players I and II each have three cards labelled $1, 2,$ and 3. Each player chooses one of her cards, independently of the other, and places it in the same envelope. If the sum of the numbers in the envelope is smaller than or equal to 4, then Player II pays Player I the sum (in £), and otherwise Player I pays Player II the sum. (For instance, if Player I chooses card 3 and Player II choose card 2, then Player I pays Player II £5.) What is the optimal strategy for each player?

Paper 4, Section II

20E Optimization

A factory produces three types of sugar, types X, Y, and Z, from three types of syrup, labelled A, B, and C. The following table contains the number of litres of syrup necessary to make each kilogram of sugar.

	X	Y	Z
A	3	2	1
B	2	3	2
C	4	1	2

For instance, one kilogram of type X sugar requires 3 litres of A, 2 litres of B, and 4 litres of C. The factory can sell each type of sugar for one pound per kilogram. Assume that the factory owner can use no more than 44 litres of A and 51 litres of B, but is required by law to use at least 12 litres of C. If her goal is to maximize profit, how many kilograms of each type of sugar should the factory produce?

Paper 1, Section I

8H Optimization

Find an optimal solution to the linear programming problem

$$\max\ 3x_1 + 2x_2 + 2x_3$$

in $x \geqslant 0$ subject to

$$\begin{aligned} 7x_1 + 3x_2 + 5x_3 &\leqslant 44, \\ x_1 + 2x_2 + x_3 &\leqslant 10, \\ x_1 + x_2 + x_3 &\geqslant 8. \end{aligned}$$

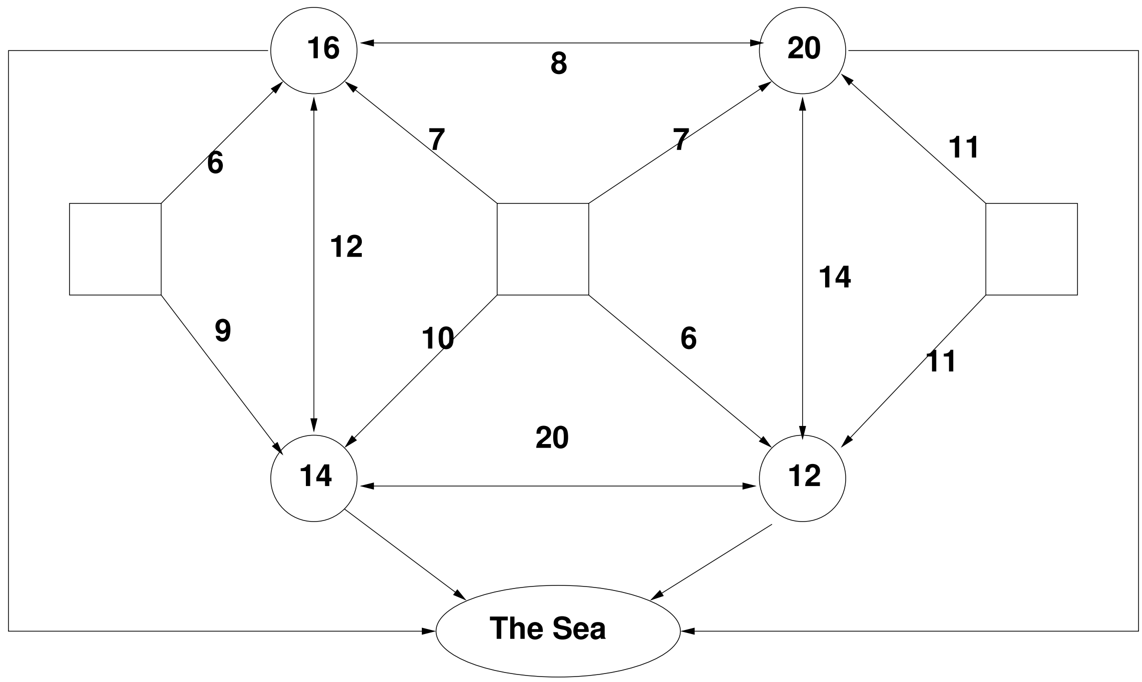
Paper 2, Section I

9H Optimization

The diagram shows a network of sewage treatment plants, shown as circles, connected by pipes. Some pipes (indicated by a line with an arrowhead at one end only) allow sewage to flow in one direction only, others (indicated by a line with an arrowhead at both ends) allow sewage to flow in either direction. The capacities of the pipes are shown. The system serves three towns, shown in the diagram as squares.

Each sewage treatment plant can treat a limited amount of sewage, indicated by the number in the circle, and this may not be exceeded for fear of environmental damage. Treated sewage is pumped into the sea, but at any treatment plant incoming untreated sewage may be immediately pumped to another plant for treatment there.

Find the maximum amount of sewage which can be handled by the system, and how this is assigned to each of the three towns.



Paper 3, Section II

20H Optimization

Four factories supply stuff to four shops. The production capacities of the factories are 7, 12, 8 and 9 units per week, and the requirements of the shops are 8 units per week each. If the costs of transporting a unit of stuff from factory i to shop j is the (i, j)th element in the matrix

$$\begin{pmatrix} 6 & 10 & 3 & 5 \\ 4 & 8 & 6 & 12 \\ 3 & 4 & 9 & 2 \\ 5 & 7 & 2 & 6 \end{pmatrix}$$

find a minimal-cost allocation of the outputs of the factories to the shops.

Suppose that the cost of producing one unit of stuff varies across the factories, being 3, 2, 4, 5 respectively. Explain how you would modify the original problem to minimise the total cost of production and of transportation, and find an optimal solution for the modified problem.

Paper 4, Section II

20H Optimization

In a pure exchange economy, there are J agents, and d goods. Agent j initially holds an endowment $x_j \in \mathbb{R}^d$ of the d different goods, $j = 1, \ldots, J$. Agent j has preferences given by a concave utility function $U_j : \mathbb{R}^d \to \mathbb{R}$ which is strictly increasing in each of its arguments, and is twice continuously differentiable. Thus agent j prefers $y \in \mathbb{R}^d$ to $x \in \mathbb{R}^d$ if and only if $U_j(y) \geqslant U_j(x)$.

The agents meet and engage in mutually beneficial trades. Thus if agent i holding z_i meets agent j holding z_j, then the amounts z_i' held by agent i and z_j' held by agent j after trading must satisfy $U_i(z_i') \geqslant U_i(z_i)$, $U_j(z_j') \geqslant U_j(z_j)$, and $z_i' + z_j' = z_i + z_j$. Meeting and trading continues until, finally, agent j holds $y_j \in \mathbb{R}^d$, where

$$\sum_j x_j = \sum_j y_j,$$

and there are no further mutually beneficial trades available to any pair of agents. Prove that there must exist a vector $v \in \mathbb{R}^d$ and positive scalars $\lambda_1, \ldots, \lambda_J$ such that

$$\nabla U_j(y_j) = \lambda_j v$$

for all j. Show that for some positive $a_1, \ldots, a_J$ the final allocations y_j are what would be achieved by a social planner, whose objective is to obtain

$$\max \sum_j a_j U_j(y_j) \quad \text{subject to} \quad \sum_j y_j = \sum_j x_j.$$

1/I/8H **Optimization**

State the Lagrangian Sufficiency Theorem for the maximization over x of $f(x)$ subject to the constraint $g(x) = b$.

For each $p > 0$, solve

$$\max \sum_{i=1}^{d} x_i^p \quad \text{subject to} \quad \sum_{i=1}^{d} x_i = 1, \quad x_i \geqslant 0\,.$$

2/I/9H **Optimization**

Goods from three warehouses have to be delivered to five shops, the cost of transporting one unit of good from warehouse i to shop j being c_{ij}, where

$$C = \begin{pmatrix} 2 & 3 & 6 & 6 & 4 \\ 7 & 6 & 1 & 1 & 5 \\ 3 & 6 & 6 & 2 & 1 \end{pmatrix}\,.$$

The requirements of the five shops are respectively 9, 6, 12, 5 and 10 units of the good, and each warehouse holds a stock of 15 units. Find a minimal-cost allocation of goods from warehouses to shops and its associated cost.

3/II/20H **Optimization**

Use the simplex algorithm to solve the problem

$$\max\ x_1 + 2x_2 - 6x_3$$

subject to $x_1, x_2 \geqslant 0$, $|x_3| \leqslant 5$, and

$$\begin{aligned} x_1 + x_2 + x_3 &\leqslant 7\,, \\ 2x_2 + x_3 &\geqslant 1\,. \end{aligned}$$

4/II/20H **Optimization**

(i) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. Suppose that the problem

$$\max\ f(x) \quad \text{subject to} \ \ g(x) = b$$

is solved by a unique $\bar{x} = \bar{x}(b)$ for each $b \in \mathbb{R}^m$, and that there exists a unique $\lambda(b) \in \mathbb{R}^m$ such that

$$\varphi(b) \equiv f(\bar{x}(b)) = \sup_x \{\, f(x) + \lambda(b)^T (b - g(x)) \,\} .$$

Assuming that $\bar{x}$ and λ are continuously differentiable, prove that

$$\frac{\partial \varphi}{\partial b_i}(b) = \lambda_i(b) . \qquad (*)$$

(ii) The output of a firm is a function of the capital K deployed, and the amount L of labour employed, given by

$$f(K, L) = K^\alpha L^\beta ,$$

where $\alpha, \beta \in (0, 1)$. The firm's manager has to optimize the output subject to the budget constraint

$$K + wL = b ,$$

where $w > 0$ is the wage rate and $b > 0$ is the available budget. By casting the problem in Lagrangian form, find the optimal solution and verify the relation $(*)$.

1/I/8C **Optimization**

State and prove the max-flow min-cut theorem for network flows.

2/I/9C **Optimization**

Consider the game with payoff matrix

$$\begin{pmatrix} 2 & 5 & 4 \\ 3 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix},$$

where the (i, j) entry is the payoff to the row player if the row player chooses row i and the column player chooses column j.

Find the value of the game and the optimal strategies for each player.

3/II/20C **Optimization**

State and prove the Lagrangian sufficiency theorem.

Solve the problem

$$\begin{array}{ll} \text{maximize} & x_1 + 3\ln(1 + x_2) \\ \text{subject to} & 2x_1 + 3x_2 \leqslant c_1, \\ & \ln(1 + x_1) \geqslant c_2, \quad x_1 \geqslant 0,\ x_2 \geqslant 0, \end{array}$$

where c_1 and c_2 are non-negative constants satisfying $c_1 + 2 \geqslant 2e^{c_2}$.

4/II/20C **Optimization**

Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & 2\,x_1 - 3\,x_2 - 2\,x_3 \\ \text{subject to} & -\,2\,x_1 + 2\,x_2 + 4\,x_3 \leqslant 5 \\ & 4\,x_1 + 2\,x_2 - 5\,x_3 \leqslant 8 \\ & 5\,x_1 - 4\,x_2 + \frac{1}{2}\,x_3 \leqslant 5\,, \qquad x_i \geqslant 0\,, \quad i = 1, 2, 3\,. \end{array}$$

(i) After adding slack variables z_1, z_2 and z_3 and performing one iteration of the simplex algorithm, the following tableau is obtained.

	x_1	x_2	x_3	z_1	z_2	z_3	
x_2	-1	1	2	1/2	0	0	5/2
z_2	6	0	-9	-1	1	0	3
z_3	1	0	17/2	2	0	1	15
Payoff	-1	0	4	3/2	0	0	15/2

Complete the solution of the problem.

(ii) Now suppose that the problem is amended so that the objective function becomes

$$2\,x_1 - 3\,x_2 - 5\,x_3\,.$$

Find the solution of this new problem.

(iii) Formulate the dual of the problem in (ii) and identify the optimal solution to the dual.

1/I/8C **Optimization**

State the Lagrangian sufficiency theorem.

Let $p \in (1, \infty)$ and let $a_1, \ldots, a_n \in \mathbb{R}$. Maximize

$$\sum_{i=1}^{n} a_i x_i$$

subject to

$$\sum_{i=1}^{n} |x_i|^p \leqslant 1, \quad x_1, \ldots, x_n \in \mathbb{R}.$$

2/I/9C **Optimization**

Consider the maximal flow problem on a finite set N, with source A, sink B and capacity constraints c_{ij} for $i, j \in N$. Explain what is meant by a cut and by the capacity of a cut.

Show that the maximal flow value cannot exceed the minimal cut capacity.

Take $N = \{0, 1, 2, 3, 4\}^2$ and suppose that, for $i = (i_1, i_2)$ and $j = (j_1, j_2)$,

$$c_{ij} = \max\{|i_1 - i_2|, |j_1 - j_2|\} \quad \text{if} \quad |i_1 - j_1| + |i_2 - j_2| = 1,$$

and $c_{ij} = 0$ otherwise. Thus the node set is a square grid of 25 points, with positive flow capacity only between nearest neighbours, and where the capacity of an edge in the grid equals the larger of the distances of its two endpoints from the diagonal. Find a maximal flow from $(0, 3)$ to $(3, 0)$. Justify your answer.

3/II/20C **Optimization**

Explain what is meant by a two-person zero-sum game with payoff matrix $A = (a_{ij} : 1 \leqslant i \leqslant m, \ \ 1 \leqslant j \leqslant n)$ and what is meant by an optimal strategy $p = (p_i : 1 \leqslant i \leqslant m)$.

Consider the following betting game between two players: each player bets an amount $1, 2, 3$ or 4; if both bets are the same, then the game is void; a bet of 1 beats a bet of 4 but otherwise the larger bet wins; the winning player collects both bets. Write down the payoff matrix A and explain why the optimal strategy $p = (p_1, p_2, p_3, p_4)^T$ must satisfy $(Ap)_i \leqslant 0$ for all i. Hence find p.

4/II/20C **Optimization**

Use a suitable version of the simplex algorithm to solve the following linear programming problem:

$$\begin{array}{lrcrcrcr} \text{maximize} & 50x_1 & - & 30x_2 & + & x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 30 \\ & 2x_1 & - & x_2 & & & \leq & 35 \\ & x_1 & + & 2x_2 & - & x_3 & \geq & 40 \\ \text{and} & & & x_1, x_2, x_3 & & & \geq & 0. \end{array}$$

1/I/8D **Optimization**

Consider the problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,\dots,m, \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,\dots,n, \\ & x_{ij} \geqslant 0, \quad \text{for all } i,j, \end{aligned}$$

where $a_i \geqslant 0$, $b_j \geqslant 0$ satisfy $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Formulate the dual of this problem and state necessary and sufficient conditions for optimality.

2/I/9D **Optimization**

Explain what is meant by a two-person zero-sum game with payoff matrix $A = (a_{ij})$.

Show that the problems of the two players may be expressed as a dual pair of linear programming problems. State without proof a set of sufficient conditions for a pair of strategies for the two players to be optimal.

3/II/20D **Optimization**

Consider the linear programming problem

$$\begin{array}{rl} \text{maximize} & 4x_1 + x_2 - 9x_3 \\ \text{subject to} & x_2 - 11x_3 \leqslant 11 \\ & -3x_1 + 2x_2 - 7x_3 \leqslant 16 \\ & 9x_1 - 2x_2 + 10x_3 \leqslant 29, \quad x_i \geqslant 0, \quad i = 1,2,3. \end{array}$$

(a) After adding slack variables z_1, z_2 and z_3 and performing one pivot in the simplex algorithm the following tableau is obtained:

	x_1	x_2	x_3	z_1	z_2	z_3	
z_1	0	1	-11	1	0	0	11
z_2	0	$\frac{4}{3}$	$-\frac{11}{3}$	0	1	$\frac{1}{3}$	$\frac{77}{3}$
x_1	1	$-\frac{2}{9}$	$\frac{10}{9}$	0	0	$\frac{1}{9}$	$\frac{29}{9}$
Payoff	0	$\frac{17}{9}$	$-\frac{121}{9}$	0	0	$-\frac{4}{9}$	$-\frac{116}{9}$

Complete the solution of the problem using the simplex algorithm.

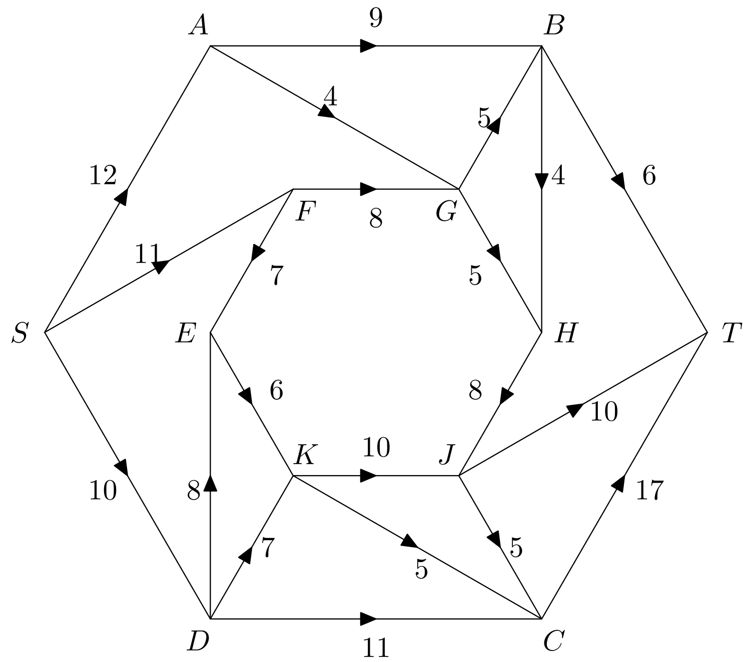
(b) Obtain the dual problem and identify its optimal solution from the optimal tableau in (a).

(c) Suppose that the right-hand sides in the constraints to the original problem are changed from $(11, 16, 29)$ to $(11 + \epsilon_1, 16 + \epsilon_2, 29 + \epsilon_3)$. Give necessary and sufficient conditions on $(\epsilon_1, \epsilon_2, \epsilon_3)$ which ensure that the optimal solution to the dual obtained in (b) remains optimal for the dual for the amended problem.

4/II/20D **Optimization**

Describe the Ford–Fulkerson algorithm for finding a maximal flow from a source to a sink in a directed network with capacity constraints on the arcs. Explain why the algorithm terminates at an optimal flow when the initial flow and the capacity constraints are rational.

Illustrate the algorithm by applying it to the problem of finding a maximal flow from S to T in the network below.



3/I/12G Optimization

Consider the two-person zero-sum game Rock, Scissors, Paper. That is, a player gets 1 point by playing Rock when the other player chooses Scissors, or by playing Scissors against Paper, or Paper against Rock; the losing player gets -1 point. Zero points are received if both players make the same move.

Suppose player one chooses Rock and Scissors (but never Paper) with probabilities p and $1-p$, $0 \leqslant p \leqslant 1$. Write down the maximization problem for player two's optimal strategy. Determine the optimal strategy for each value of p.

3/II/23G Optimization

Consider the following linear programming problem:

$$\text{maximize } -x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \qquad x_1 + x_2 &\geqslant 3, \\ -x_1 + 2x_2 &\geqslant 6, \\ -x_1 + x_2 &\leqslant 2, \\ x_2 &\leqslant 5, \\ x_i &\geqslant 0, \quad i = 1, 2. \end{aligned}$$

Write down the Phase One problem in this case, and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve the above maximization problem. That is, find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

4/I/10G Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow; maximal flow; cut; capacity.

4/II/20G Optimization

For any number $c \in (0,1)$, find the minimum and maximum values of

$$\sum_{i=1}^{n} x_i^c \,,$$

subject to $\sum_{i=1}^{n} x_i = 1, x_1, \ldots, x_n \geqslant 0$. Find all the points $(x_1, \ldots, x_n)$ at which the minimum and maximum are attained. Justify your answer.

3/I/5H Optimization

Two players A and B play a zero-sum game with the pay-off matrix

	B_1	B_2	B_3
A_1	4	−2	−5
A_2	−2	4	3
A_3	−3	6	2
A_4	3	−8	−6

Here, the (i, j) entry of the matrix indicates the pay-off to player A if he chooses move A_i and player B chooses move B_j. Show that the game can be reduced to a zero-sum game with 2×2 pay-off matrix.

Determine the value of the game and the optimal strategy for player A.

3/II/15H Optimization

Explain what is meant by a transportation problem where the total demand equals the total supply. Write the Lagrangian and describe an algorithm for solving such a problem. Starting from the north-west initial assignment, solve the problem with three sources and three destinations described by the table

5	9	1	36
3	10	6	84
7	2	5	40
14	68	78	

where the figures in the 3×3 box denote the transportation costs (per unit), the right-hand column denotes supplies, and the bottom row demands.

4/I/5H Optimization

State and prove the Lagrangian sufficiency theorem for a general optimization problem with constraints.

4/II/14H Optimization

Use the two-phase simplex method to solve the problem

$$\begin{array}{lrl} \text{minimize} & 5x_1 - 12x_2 + 13x_3 & \\ \text{subject to} & 4x_1 + 5x_2 & \leq 9, \\ & 6x_1 + 4x_2 + x_3 & \geq 12, \\ & 3x_1 + 2x_2 - x_3 & \leq 3, \\ & x_i \geq 0, \quad i = 1, 2, 3. & \end{array}$$

3/I/5H Optimization

Consider a two-person zero-sum game with a payoff matrix

$$\begin{pmatrix} 3 & b \\ 5 & 2 \end{pmatrix},$$

where $0 < b < \infty$. Here, the (i, j) entry of the matrix indicates the payoff to player one if he chooses move i and player two move j. Suppose player one chooses moves 1 and 2 with probabilities p and $1 - p$, $0 \leq p \leq 1$. Write down the maximization problem for the optimal strategy and solve it for each value of b.

3/II/15H Optimization

Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 1, \\ & 4x_1 - x_2 \geq 10, \\ & x_2 \leq 6, \\ & x_i \geq 0,\ i = 1, 2. \end{array} \qquad (1)$$

Write down the Phase One problem for (1) and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve (1), i.e., find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

4/I/5H Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow, maximal flow, cut, capacity.

4/II/14H **Optimization**

A gambler at a horse race has an amount $b > 0$ to bet. The gambler assesses p_i, the probability that horse i will win, and knows that $s_i \geq 0$ has been bet on horse i by others, for $i = 1, 2, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_n)$ so that it maximizes

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0, \tag{1}$$

where x_i is the amount the gambler bets on horse i. Show that the optimal solution to (1) also solves the following problem:

$$\text{minimize } \sum_{i=1}^{n} \frac{p_i s_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0.$$

Assume that $p_1/s_1 \geq p_2/s_2 \geq \ldots \geq p_n/s_n$. Applying the Lagrangian sufficiency theorem, prove that the optimal solution to (1) satisfies

$$\frac{p_1 s_1}{(s_1 + x_1)^2} = \ldots = \frac{p_k s_k}{(s_k + x_k)^2}, \quad x_{k+1} = \ldots = x_n = 0,$$

with maximal possible $k \in \{1, 2, \ldots, n\}$.

[*You may use the fact that for all $\lambda < 0$, the minimum of the function $x \mapsto \frac{ps}{s+x} - \lambda x$ on the non-negative axis $0 \leq x < \infty$ is attained at*

$$x(\lambda) = \left(\sqrt{\frac{ps}{-\lambda}} - s\right)^{+} \equiv \max\left(\sqrt{\frac{ps}{-\lambda}} - s, 0\right).]$$

Deduce that if b is small enough, the gambler's optimal strategy is to bet on the horses for which the ratio p_i/s_i is maximal. What is his expected gain in this case?

3/I/5D **Optimization**

Let $a_1, \ldots, a_n$ be given constants, not all equal.

Use the Lagrangian sufficiency theorem, which you should state clearly, without proof, to minimize $\sum_{i=1}^n x_i^2$ subject to the two constraints $\sum_{i=1}^n x_i = 1, \sum_{i=1}^n a_i x_i = 0$.

3/II/15D **Optimization**

Consider the following linear programming problem,

$$\begin{array}{ll} \text{minimize} & (3-p)x_1 + px_2 \\ \text{subject to} & \begin{array}{rl} 2x_1 + x_2 & \geqslant 8 \\ x_1 + 3x_2 & \geqslant 9 \\ x_1 \qquad & \leqslant 6 \\ x_1,\ x_2 \quad & \geqslant 0. \end{array} \end{array}$$

Formulate the problem in a suitable way for solution by the two-phase simplex method.

Using the two-phase simplex method, show that if $2 \leqslant p \leqslant \frac{9}{4}$ then the optimal solution has objective function value $9 - p$, while if $\frac{9}{4} < p \leqslant 3$ the optimal objective function value is $18 - 5p$.

4/I/5D **Optimization**

Explain what is meant by a two-person zero-sum game with payoff matrix $A = [a_{ij}]$. Write down a set of sufficient conditions for a pair of strategies to be optimal for such a game.

A fair coin is tossed and the result is shown to player I, who must then decide to 'pass' or 'bet'. If he passes, he must pay player II £1. If he bets, player II, who does not know the result of the coin toss, may either 'fold' or 'call the bet'. If player II folds, she pays player I £1. If she calls the bet and the toss was a head, she pays player I £2; if she calls the bet and the toss was a tail, player I must pay her £2.

Formulate this as a two-person zero-sum game and find optimal strategies for the two players. Show that the game has value $\frac{1}{3}$.

[*Hint: Player I has four possible moves and player II two.*]

4/II/14D **Optimization**

Dumbledore Publishers must decide how many copies of the best-selling "History of Hogwarts" to print in the next two months to meet demand. It is known that the demands will be for 40 thousand and 60 thousand copies in the first and second months respectively, and these demands must be met on time. At the beginning of the first month, a supply of 10 thousand copies is available, from existing stock. During each month, Dumbledore can produce up to 40 thousand copies, at a cost of 400 galleons per thousand copies. By having employees work overtime, up to 150 thousand additional copies can be printed each month, at a cost of 450 galleons per thousand copies. At the end of each month, after production and the current month's demand has been satisfied, a holding cost of 20 galleons per thousand copies is incurred.

Formulate a transportation problem, with 5 supply points and 3 demand points, to minimize the sum of production and holding costs during the two month period, and solve it.

[*You may assume that copies produced during a month can be used to meet demand in that month.*]