# 11 Minimum Cost Circulation Problems

## 11.1 Sufficient conditions for a minimal cost circulation

Recall the minimum cost circulation problem:

$$\text{minimize} \sum_{ij} d_{ij} x_{ij}$$
$$\text{subject to} \sum_j x_{ij} - \sum_j x_{ji} = 0, \text{ each } i, \text{ and } c^-_{ij} \le x_{ij} \le c^+_{ij}.$$

Consider the Lagrangian for the problem of finding the minimum cost circulation. We shall treat the capacity constraints as the region constraints, so

$$X = \{x_{ij} : c^-_{ij} \le x_{ij} \le c^+_{ij}\}.$$

We introduce Lagrange multipliers $\lambda_i$ (one for each node) and write

$$L(x,\lambda) = \sum_{ij} d_{ij} x_{ij} - \sum_i \lambda_i \left( \sum_j x_{ij} - \sum_j x_{ji} \right)$$

Rearranging we obtain

$$L(x,\lambda) = \sum_{ij} (d_{ij} - \lambda_i + \lambda_j) x_{ij}.$$

We attempt to minimize $L(x,\lambda)$ in $X$.

Provided $c^-_{ij}, c^+_{ij}$ are finite we see that there is a finite minimum for all $\lambda$, achieved such that

$$x_{ij} = \begin{cases} c^-_{ij} & \text{if} \quad d_{ij} - \lambda_i + \lambda_j > 0 \\ c^+_{ij} & \text{if} \quad d_{ij} - \lambda_i + \lambda_j < 0 \end{cases} \tag{1}$$

$$c^-_{ij} \le x_{ij} \le c^+_{ij} \quad \text{if} \quad d_{ij} - \lambda_i + \lambda_j = 0. \tag{2}$$

**Theorem 11.1** *If $(x_{ij})$ is a feasible circulation and there exists $\lambda$ such that $(x_{ij}), \lambda$ satisfy conditions (1) and (2) above, then $(x_{ij})$ is a minimal cost circulation.*

Proof. Apply the Lagrangian sufficiency theorem. ■

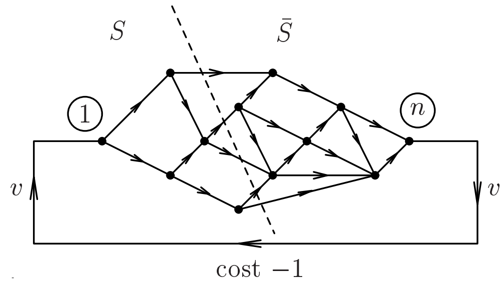
**Definition 11.1** *The Lagrange multipliers $\lambda_i$ are usually known as* **node numbers** *or* **potentials** *in network problems.*

**Definition 11.2** *$\lambda_i - \lambda_j$ is known as the* **tension** *in the arc $(i,j)$.*



## 11.2 Max-flow as a minimum cost circulation problem

The maximal flow problem we studied earlier can be set up as a minimal cost circulation problem. For each arc in the network we assign a capacity constraint $0 \le x_{ij} \le c^+_{ij}$ and all cost $d_{ij} = 0$. Add an arc from node $n$ to 1 with no capacity constraint and cost $-1$.




The cost of the circulation is $-v$, so minimizing $-v$ is the same as maximizing $v$. Let us seek node numbers $\lambda_i$ which will satisfy optimality for this problem. Since arc $(n,1)$ has no capacity constraints, for a finite optimum we will require

$$d_{n1} - \lambda_n + \lambda_1 = 0 \Longrightarrow \lambda_1 = \lambda_n + 1.$$

Let us set $\lambda_n = 0$, $\lambda_1 = 1$. (Since it is only the differences in the $\lambda$s that matter, we can pick one arbitrarily.) Let $(S, \bar{S})$ be a minimal cut. Assign $\lambda_i = 1$ for $i \in S$ and $\lambda_i = 0$ for $i \in \bar{S}$. Now check that

(a) For $i, j \in S$ or $i, j \in \bar{S} \Longrightarrow d_{ij} - \lambda_i + \lambda_j = 0$ so $x_{ij}$ can take any feasible value.

(b) For $i \in S$, $j \in \bar{S}$ we have

$$d_{ij} - \lambda_i + \lambda_j = 0 - 1 + 0 = -1 \Longrightarrow x_{ij} = c^+_{ij}.$$

(c) For $i \in \bar{S}$, $j \in S$ we have

$$d_{ij} - \lambda_i + \lambda_j = 0 - 0 + 1 = 1 \Longrightarrow x_{ij} = 0.$$

But conditions (a)–(c) are precisely those satisfied by a maximal flow and minimal cut.

If we like, we can say that the Ford-Fulkerson algorithm in looking for a cut is trying to find node numbers and a flow to satisfy optimality conditions.

**Remark**

In many problems it is natural to take $c^-_{ij} = 0$, $c^+_{ij} = \infty$. In this case we will achieve a finite optimum only if $d_{ij} - \lambda_i + \lambda_j \ge 0$ for each arc.

**Theorem 11.2** *For a minimal cost circulation problem with capacity constraints* $0 \leq x_{ij} < \infty$ *on each arc* $(i,j)$*, if we have a feasible circulation* $(x_{ij})$ *and node numbers* $\lambda_i$ *such that*

$$\begin{gathered} d_{ij} - \lambda_i + \lambda_j \geq 0, \text{ each } (i,j), \text{ and} \\ x_{ij} = 0 \text{ if } d_{ij} - \lambda_i + \lambda_j > 0, \end{gathered}$$

*then* $(x_{ij})$ *is optimal.*

Proof. Apply the Lagrangian sufficiency theorem. ■

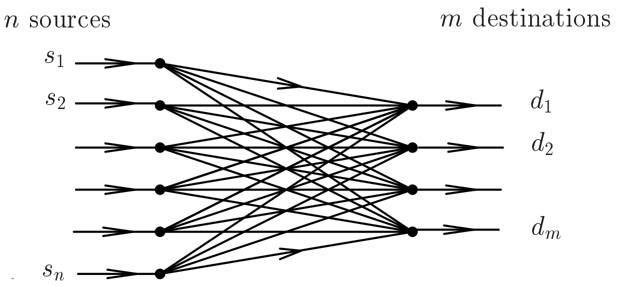
Note: The optimality conditions imply $(d_{ij} - \lambda_i + \lambda_j)x_{ij} = 0$ in this case (complementary slackness).

## 11.3 The transportation problem

Consider a network representing the problem of a supplier who has $n$ supply depots from which goods must be shipped to $m$ destinations. We assume there are quantities $s_1, \ldots, s_n$ of the goods at depots $\{S_1, \ldots, S_n\}$ and that the demands at destinations $\{D_1, \ldots, D_m\}$ are given by $d_1, \ldots, d_m$. We also assume that $\sum_i s_i = \sum_j d_j$ so that total supply = total demand. Any amount of goods may be taken directly from source $i$ to destination $j$ at a cost of $d_{ij}$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ per unit. One formulation of the problem is

$$\begin{gathered} \text{minimize } \sum_{ij} d_{ij} x_{ij} \\ \text{subject to } \sum_j x_{ij} = s_i \text{ each } i, \quad \sum_i x_{ij} = d_j \text{ each } j \\ \text{with } x_{ij} \geq 0 \text{ each } i, j \end{gathered}$$

Here $x_{ij}$ is the flow from $S_i$ to $D_j$. The network looks like:




with arcs $(i,j)$, $0 \leq x_{ij} < \infty$, and cost $d_{ij}$ per unit flow.



The Lagrangian for the problem can be written

$$L(x, \lambda, \mu) = \sum_{ij} d_{ij} x_{ij} - \sum_i \lambda_i \left( \sum_j x_{ij} - s_i \right) + \sum_j \mu_j \left( \sum_i x_{ij} - d_j \right),$$

where we label Lagrange multipliers (node numbers) $\lambda_i$ for sources and $\mu_j$ for destinations. (We choose the sign of $\mu_j$ in this apparently unusual way to be consistent with our previous treatment. The demands $d_j$ at the destination are really negative supplies.) Rearranging,

$$L(x, \lambda, \mu) = \sum_{ij} (d_{ij} - \lambda_i + \mu_j) x_{ij} + \sum_i \lambda_i s_i - \sum_j \mu_j d_j.$$

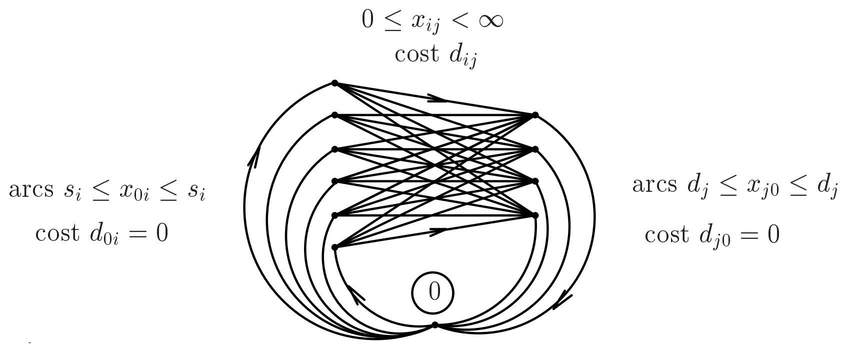
This will have a finite minimum in $x_{ij} \geq 0$, and the minimum occurs with $(d_{ij} - \lambda_i + \mu_j)x_{ij} = 0$ on each arc. Thus the Lagrangian sufficiency theorem give the same optimality conditions as before.

**Theorem 11.3** *A flow* $x_{ij}$ *is optimal for the transportation problem if* $\exists\ \lambda_i, \mu_j$ *such that* $d_{ij} - \lambda_i + \mu_j \geq 0$ *each* $(i,j)$ *and* $(d_{ij} - \lambda_i + \mu_j)x_{ij} = 0$.

Proof. The Lagrangian sufficiency theorem applies. ■

**Remark**

It is no surprise that the same optimality conditions appear as in the minimal cost circulation problem. If we augment the transportation network by connecting all sources and all destinations to a common 'artificial node' by arcs where the flow is constrained to be exactly that which is required (and zero cost) we obtain the same problem as in minimal cost circulation form.




The optimality conditions on the extra arcs are automatically satisfied by a feasible flow since $x_{j0} = d_j$, $x_{0i} = s_i$ regardless of node numbers.