2 The Solution of LP Problems

2.1 Basic solutions

Returning to P. We need a more algebraic (less geometric) characterisation of extreme points. Let us rewrite P with equality constraints, using slack variables.

$$\begin{array}{rrcl} \text{P:} & \text{maximize} & x_1 + x_2 & \\ & \text{subject to} & x_1 + 2x_2 + z_1 & = 6 \\ & & x_1 - x_2 + z_2 & = 3 \\ & & x_1, x_2, z_1, z_2 & \geq 0 \end{array}$$

Let us calculate the value of the variables at each of the 6 points marked A–F in our picture of the feasible set for P. The values are:

	x_1	x_2	z_1	z_2	f
A	0	0	6	3	0
B	3	0	3	0	3
C	4	1	0	0	5
D	0	3	0	6	3
E	6	0	0	−3	6
F	0	−3	12	0	−3

At each point there are two zero and two non-zero variables. This is not surprising.

Geometrically: The 4 lines defining the feasible set can be written $x_1 = 0$; $x_2 = 0$; $z_1 = 0$; $z_2 = 0$. At the intersection of each pair of lines, two variables are zero.

Algebraically: Constraints $Ax + z = b$ are 2 equations in 4 unknowns. If we choose 2 variables (which can be done in $\binom{4}{2} = 6$ ways) and set them equal to zero we will be left with two equations in the other two variables. So (provided A and b are 'nice') there will be a unique solution for the two non-zero variables.

Instead of calling the slack variables z_1 and z_2, let us call them x_3 and x_4 so that we can write P as

$$\begin{array}{rrcl} \text{P:} & \text{maximize} & x_1 + x_2 & \\ & \text{subject to} & Ax & = b \\ & & x & \geq 0 \end{array}$$

Note we have to extend A to $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ and x to $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. In this version of P, A is $(m \times n)$ with $n > m$ and there are m equations in $n > m$ unknowns. We

can therefore choose $(n - m)$ variables in $\binom{n}{m}$ ways. Set them to zero. There will be a unique solution to $Ax = b$ for the remaining m variables (provided A and b are 'nice').

Definition 2.1 *A* **basic solution** *to $Ax = b$ is a solution with at least $n - m$ zero variables. The solution is a* **non-degenerate** *basic solution if exactly $n - m$ variables are zero. The choice of the m non-zero variables is called the* **basis**. *Variables in the basis are called* **basic**; *the others are called* **non-basic**.

Definition 2.2 *If a basic solution satisfies $x \geq 0$ then it is called a* **basic feasible solution**.

So A–F are basic solutions (and non-degenerate) and A–D are basic feasible solutions.

Theorem 2.1 *Basic feasible solutions $\equiv$ extreme points of the feasible set.*

Taking Theorems 1.2 and 2.1 together,

Theorem 2.2 *If an LP has a finite solution then there is an optimal basic feasible solution.*

So we can do algebra instead of drawing a picture (which is good for computer and good for us if there are many variables and constraints.) Our previous (and foolish) algorithm can be restated:

Algorithm:

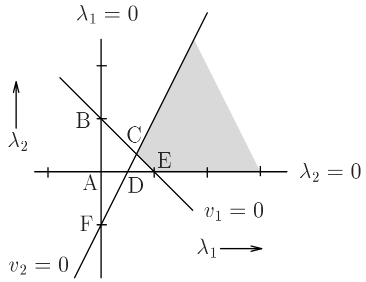
1. Find all the basic solutions.
2. Test to see which are feasible.
3. Choose the best basic feasible solution.

Unfortunately it is not usually easy to know which basic solutions will turn out to be feasible before calculating them. Hence, even though there are often considerably fewer basic feasible solutions we will still need to calculate all $\binom{n}{m}$ basic solutions.

2.2 Primal-dual relationships

Now let us look at problem D which can be written, after introducing slack variables v_1 and v_2 as

$$\begin{array}{rrcl} \text{D:} & \text{minimize} & 6\lambda_1 + 3\lambda_2 & \\ & \text{subject to} & \lambda_1 + \lambda_2 - v_1 & = 1 \\ & & 2\lambda_1 - \lambda_2 - v_2 & = 1 \\ & & \lambda_1, \lambda_2, v_1, v_2 & \geq 0 \end{array}$$



The value of the variables, etc., at the points A–F in P (as above) and D are:

P:

	x_1	x_2	z_1	z_2	f
A	0	0	6	3	0
B	3	0	3	0	3
C	4	1	0	0	5
D	0	3	0	6	3
E	6	0	0	-3	6
F	0	-3	12	0	-3

D:

	v_1	v_2	λ_1	λ_2	f
A	-1	-1	0	0	0
B	0	-2	0	1	3
C	0	0	$\frac{2}{3}$	$\frac{1}{3}$	5
D	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	3
E	0	1	1	0	6
F	-2	0	0	-1	-3

Observe, that for D, as for P above, there are two zero and two non-zero variables at each intersection (basic solution). C and E are feasible for D. The optimum is at C with optimum value 5 (assuming we are minimizing and the other basic solutions are not feasible.)

We make the following observations by comparing lists of basic solutions for P and D. (Greater explanation is given later.)

1. For each solution for P there is a corresponding basic solution for D. [Labels A–F have been chosen so that corresponding solutions have the same labels.] Each pair

   (a) has the same value of the objective function.

   (b) satisfies **complementary slackness**, i.e., $x_i v_i = 0$, $\lambda_i z_i = 0$,

   so for each corresponding pair,

P		D
variables x		constraints
x_i basic ($x_i \neq 0$)	$\Longleftrightarrow$	constraint: tight ($v_i = 0$)
x_i non-basic ($x_i = 0$)	$\Longleftrightarrow$	constraint: slack ($v_i \neq 0$)
constraints		variables λ
constraint: tight ($z_i = 0$)	$\Longleftrightarrow$	λ_i basic ($\lambda_i \neq 0$)
constraint: slack ($z_i \neq 0$)	$\Longleftrightarrow$	λ_i non-basic ($\lambda_i = 0$)

   (These conditions determine which basic solutions in P and D are paired; the implications go both ways because in this example all basic solutions are non-degenerate.)

2. There is only one pair that is feasible for both P and D, and that solution is C, which is optimal, with value 5, for both.

3. For any x feasible for P and λ feasible for D we have $c^\top x \le b^\top \lambda$ with equality if and only if x, λ are optima for P and D.

This correspondence between P and D is so symmetrical (and pretty) that it feels as though it ought to be obvious why it works. We will do the necessary proofs for most of it later, but be reassured that it is not all obvious. In particular, it is hard to prove that P and D have the same solution. But let us write down the theorems.

Let P (primal) and D (dual) be the LPs

$$\text{P: } \max\ c^\top x \quad \text{s.t. } Ax \le b,\ x \ge 0,$$
$$\text{D: } \min\ b^\top \lambda \quad \text{s.t. } A^\top \lambda \ge c,\ \lambda \ge 0.$$

Then

Theorem 2.3 (weak duality) *If x is feasible for P and λ is feasible for D then $c^\top x \le b^\top \lambda$. (In particular, if one problem is feasible then the other is bounded.)*

Theorem 2.4 (sufficient conditions for optimality) *If x is feasible for P and λ is feasible for D, and x, λ satisfy complementary slackness, then x is optimal for P and λ is optimal for D. Furthermore $c^\top x = \lambda^\top b$.*

Theorem 2.5 (strong duality: necessary conditions for optimality) *If both P and D are feasible (each has at least one feasible solution), then $\exists\ x, \lambda$ satisfying the conditions of Theorem 2.4 above.*

Remarks.

1. We will prove Theorems 2.3 and 2.4 in quite a general way. Theorem 2.5 is harder and we will only sketch the general proof.

2. Note that if we write D as

$$\max\ -b^\top \lambda \quad \text{s.t. } -A^\top \lambda \le -c,\ \lambda \ge 0,$$

   and now construct the dual of D (in the same way as we obtained D from P), we recover P. For LP problems *the dual of the dual is always the primal.*

3. Theorems 2.3–2.5, and the whole idea of constructing the dual problem, work in quite general convex optimization problems (not just LP problems). We shall see some of this. It is a pretty theory.

4. Why do we care about the dual problem instead of just getting on with the solution of P?

    (a) The statement of optimality conditions in terms of P and D in Theorem 2.4 is powerful and natural.

    (b) It is sometimes easier to solve D than P (and they have the same solution).

    (c) For some problems it is natural to consider both P and D together (e.g., two person zero-sum games).

    (d) The theory is helpful in constructing algorithms. Theorem 2.4 says that for optimality we need three things: primal feasibility, dual feasibility and complementary slackness. Some algorithms start with solutions that are primal feasible and work towards dual feasibility. Others start with dual feasibility. Yet others (in particular network problems) alternately look at the primal and dual variables and move towards feasibility for both at once.