9 Two Person Zero-Sum Games

9.1 Games with a saddle-point

We consider games that are zero-sum, in the sense that one player wins what the other loses. The players make moves simultaneously. Each has a choice of moves (not necessarily the same). There is a **pay-off matrix** $A = (a_{ij})$.

$$
\begin{array}{cc}
 & \text{II plays } j \\
 & \begin{array}{cccc} \;\;1 & \;\;\;2 & \;\;\;3 & \;\;\;4 \end{array} \\
\text{I plays } i \; \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left|\begin{array}{rrrr} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & 6 & 0 & -5 \end{array}\right| \leftarrow (a_{ij})
\end{array}
$$

If player I makes move i and player II makes move j then player I wins (and player II loses) a_{ij}. What is the best strategy for the two players to adopt? We assume that they both know the matrix.

Let us ask what is the best that player I can do if player II plays move j.

II's move: $j =$	1	2	3	4	
I's best response: $i =$	2	3	2	1	
I wins	5	6	4	20	$\leftarrow$ column maximums

Similarly, we ask what is the best that player II can do if I plays move i?

I's move: $i =$	1	2	3	
II's best response: $j =$	1	3	4	
I wins	-5	4	-5	$\leftarrow$ row minimums

Here the minimal column maximum = maximal row minimum = 4, when player I plays 2 and player II plays 3. In this case we say that A has a **saddle-point** $(2, 3)$ and the game is solved.

Remarks. The game is solved in the sense that

1. Each player maximizes his minimum gain.

2. If either player announces any strategy (in advance) other than 'I plays 2' and 'II plays 3', he will do worse (including the announcement of a mixed strategy — when he plays $1, \dots, n$ with probabilities $p_1, \dots, p_n$).

3. Even if either player announces that he will play the saddle-point move is advance, the other player cannot improve on the saddle-point.

9.2 Example: Two-finger Morra, a game without a saddle-point

Each player displays either one or two fingers and simultaneously guesses how many fingers his opponent will show. If both players guess correctly or both guess incorrectly then the game is a tie. If only one player guesses correctly, then that player wins from the other player an amount equal to the total number of fingers shown. A strategy for a player is $(a, b) =$'show a, guess b'. The pay-off matrix is

$$\begin{array}{c|cccc|}
 & (1{,}1) & (1{,}2) & (2{,}1) & (2{,}2) \\
\hline
(1{,}1) & 0 & 2 & -3 & 0 \\
(1{,}2) & -2 & 0 & 0 & 3 \\
(2{,}1) & 3 & 0 & 0 & -4 \\
(2{,}2) & 0 & -3 & 4 & 0 \\
\hline
\end{array} = (a_{ij})$$

Column maximums are all positive and row minimums are all negative. So there is no saddle point (even though the game is symmetric and fair). If either player announces a fixed strategy (in advance), the other player will win. We assume each player will choose a mixed strategy where he plays move i with probability p_i and that each player attempts to maximize his minimum gain.

9.3 Determination of an optimal strategy

Each player must use a **mixed strategy**. We assume player I moves i with probability p_i, $i = 1, \dots, n$ and player II moves j with probability q_j, $j = 1, \dots, m$. Player I's expected payoff if player II plays move j is

$$\sum_i a_{ij} p_i.$$

So player I attempts to

$$\text{maximize} \left\{ \min_j \sum_i a_{ij} p_i \right\} \quad \text{s.t.} \quad \sum_i p_i = 1, \; p_i \geq 0.$$

Note that this is equivalent to

$$\text{P: } \max v \quad \text{s.t.} \quad \sum_i a_{ij} p_i \geq v, \text{ each } j, \text{ and } \sum_i p_i = 1, \; p_i \geq 0,$$

since v on being maximized will increase until it equals the minimum of the $\sum_i a_{ij} p_i$. By similar arguments, player II's problem is

$$\text{D: } \min v \quad \text{s.t.} \quad \sum_j a_{ij} q_j \leq v, \text{ each } i, \text{ and } \sum_j q_j = 1, \; q_j \geq 0.$$

It is possible to show that P and D are duals to one another (by the standard technique of finding the dual of P). Consequently, the general theory gives sufficient conditions for strategies p and q to be optimal.

Theorem 9.1 *If v, p and q, with $\sum_i p_i = \sum_j q_j = 1$, $p_i, q_j \geq 0$ are such that $p^\top A \geq v$ and $Aq \leq v$ and $v = p^\top Aq$ then p is optimal for P and q is optimal for D with common optimum (the* **value of the game***) v.*

Proof. $p^\top A \geq v$ (which is understood to mean $(p^\top A)_i \geq v$ in each component i) is just primal feasibility. Similarly, $Aq \leq v$ is dual feasibility. In addition,

$$
\begin{aligned}
v = p^\top Aq \quad &\Longrightarrow \quad v = \sum_i p_i \sum_j a_{ij} q_j \\
&\Longrightarrow 0 = \sum_i p_i \left(\sum_j a_{ij} q_j - v \right).
\end{aligned}
$$

Similarly,

$$0 = \sum_j q_j \left(\sum_i p_i a_{ij} - v \right).$$

These are the complementary slackness conditions for this pair of problems. Thus p and q are optimal for P and D respectively. Clearly each has optimum v. ■

Remarks.

1. These conditions allow us to check optimality. For small problems one can often use them to find the optimal strategies, but for larger problems it will be best to use some other technique to find the optimum (e.g., simplex algorithm). Note, however, that the problems P and D are not in a form where we can apply the simplex algorithm directly; the p_is satisfy $p_i \geq 0$, but v does not have a positivity constraint. Also the constraints are $\sum_i a_{ij} p_i - v = 0$ with r.h.s.$= 0$. It is possible, however, to transform the problem into a form amenable to the simplex algorithm.

   (a) Add a constant k to each a_{ij} so that $a_{ij} > 0$ each i, j. This doesn't change anything about the game except the value which is now guaranteed to be positive ($v > 0$).

   (b) Change variables to $x_i = p_i / v$. We now have

   $$\max v \quad \text{s.t.} \ \sum_i a_{ij} x_i \geq 1, \ \sum_i x_i = 1/v, \ x_i \geq 0,$$

   which is equivalent to

   $$\min \sum_i x_i \quad \text{s.t.} \ \sum_i a_{ij} x_i \geq 1, \ x_i \geq 0$$

   and this is the type of problem we are used to.

2. Check that the general theorem above gives the right answer for a game with a saddle-point (a, b), (i.e., $p_a = 1$, $q_b = 1$ and other $p_i, q_j = 0$).

3. Two-finger Morra has an optimal solution $p = q = (0, \frac{3}{5}, \frac{2}{5}, 0)$, $v = 0$, as can be easily checked. (But it is not uniquely optimal. Another optimal solution is $p = q = (0, \frac{4}{7}, \frac{3}{7}, 0)$.) It is obvious that we expect to have $p = q$ and $v = 0$ since the game is symmetric between the players ($A = -A^\top$). A is called an **anti-symmetric matrix**.

9.4 Example: Colonel Blotto

Colonel Blotto has three regiments and his enemy has two regiments. Both commanders are to divide their regiments between two posts. At each post the commander with the greater number of regiments wins one for each conquered regiment, plus one for the post. If the commanders allocate equal numbers of regiments to a post there is a stand-off. This gives the pay-off matrix

		Enemy commander (2,0)	(1,1)	(0,2)
Colonel Blotto	(3,0)	3	1	0
	(2,1)	1	2	-1
	(1,2)	-1	2	1
	(0,3)	0	1	3

Clearly it is optimal for Colonel Blotto to divide his regiments (i, j) and (j, i) with equal probability. So the game reduces to one with the payoff matrix

	(2,0)	(1,1)	(0,2)
(3,0) or (0,3)	$\frac{3}{2}$	1	$\frac{3}{2}$
(2,1) or (1,2)	0	2	0

You can try deriving the optimal solution by

(a) looking at player Colonel Blotto's original problem: maximize $\{\min_j \sum_i p_i a_{ij}\}$, i.e., $\text{maximize}_p \min\{\frac{3}{2}p, p + 2(1-p)\}$,

(b) attempting to derive p, q, v from the conditions $p^\top A \geq v$, $Aq \leq v$, $p^\top Aq = v$, or

(c) converting the problem as explained above and using the simplex method.

For this game, $p = (\frac{4}{5}, \frac{1}{5})$, $q = (\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$ and $v = \frac{6}{5}$ is optimal. In the the original problem, this means that Colonel Blotto should distribute his regiments as (3,0), (2,1), (1,2), (0,3) with probabilities $\frac{2}{5}, \frac{1}{10}, \frac{1}{10}, \frac{2}{5}$ respectively, and his enemy should distribute hers as (2,0), (1,1), (0,2) with probabilities $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$ respectively.