4 The Simplex Tableau

4.1 Choice of pivot column

We might have chosen the first pivot as a_{12} which would have resulted in

$$
\begin{array}{r|cccc|c}
 & x_1 & x_2 & z_1 & z_2 & a_{i0} \\
\hline
z_1 \text{ basic} & 1 & 2 & 1 & 0 & 6 \\
z_2 \text{ basic} & 1 & -1 & 0 & 1 & 3 \\
\hline
a_{0j} & 1 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

$$
\longrightarrow
\begin{array}{r|cccc|c}
 & x_1 & x_2 & z_1 & z_2 & a_{i0} \\
\hline
x_2 \text{ basic} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 3 \\
z_2 \text{ basic} & \frac{3}{2} & 0 & \frac{1}{2} & 1 & 6 \\
\hline
a_{0j} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -3 \\
\hline
\end{array}
$$

This is the tableau for vertex D. A further iteration, with pivot a_{21} takes us to the optimal solution at vertex C. Therefore both choices of initial pivot column resulted in it requiring two steps to reach the optimum.

Remarks.

1. In general, there is no way to tell in advance which choice of pivot column will result in the smallest number of iterations. We may choose any column where $a_{0j} > 0$. A common rule-of-thumb is to choose the column for which a_{0j} is greatest, since the objective function increases by the greatest amount per unit increase in the variable corresponding to that column.

2. At each stage of the simplex algorithm we have two things in mind.

   First — a particular choice of basis and basic solution.

   Second — a rewriting of the problem in a convenient form.

   There is always an identity matrix embedded in the tableau corresponding to the basic variables. Hence, when the non-basic variables are set to zero the equations are trivial to solve for the values of the basic variables. They are just given by the right-hand column.

   Check that provided we start with the equations written in this form in the initial tableau, the simplex algorithm rules ensure that we obtain an identity matrix at each stage.

3. The tableau obviously contains some redundant information. For example, provided we keep track of which equation corresponds to a basic variable, we could omit the columns corresponding to the identity matrix (and zeros in the objective row). This is good for computer programs, but it is probably better to keep the whole thing for hand calculation.

4.2 Initialization: the two-phase method

In our example there was an obvious basic feasible solution with which to start the simplex algorithm. This is not always the case. For example, suppose we have a problem like

$$\begin{array}{lrcl}
\text{maximize} & -6x_1 - 3x_2 & & \\
\text{subject to} & x_1 + x_2 & \geq & 1 \\
& 2x_1 - x_2 & \geq & 1 \\
& 3x_2 & \leq & 2 \\
& x_1, x_2 & \geq & 0
\end{array}$$

which we wish to solve using the simplex algorithm. We can add slack variables (sometimes called **surplus variables** when they appear in $\geq$ constraints) to obtain

$$\begin{array}{lrcl}
\text{maximize} & -6x_1 - 3x_2 & & \\
\text{subject to} & x_1 + x_2 - z_1 & = & 1 \\
& 2x_1 - x_2 - z_2 & = & 1 \\
& 3x_2 + z_3 & = & 2 \\
& x_i, z_i & \geq & 0
\end{array}$$

but there is no obvious b.f.s. since $z_1 = -1$, $z_2 = -1$, $z_3 = 2$ is not feasible.

The trick is to add extra variables called **artificial variables**, y_1, y_2 so that the constraints are

$$\begin{array}{rcl}
x_1 + x_2 - z_1 + y_1 & = & 1 \\
2x_1 - x_2 - z_2 + y_2 & = & 1 \\
3x_2 + z_3 & = & 2 \\
x_i, z_i, y_i & \geq & 0
\end{array}$$

Phase I is to minimize $y_1 + y_2$ and we can start this phase with $y_1 = 1$, $y_2 = 1$ and $z_3 = 2$. (Notice we did not need an artificial variable in the third equation.) Provided the original problem is feasible we should be able to obtain a minimum of 0 with $y_1 = y_2 = 0$ (since y_1 and y_2 are not needed to satisfy the constraints if the original problem is feasible). The point of this is that at the end of Phase I the simplex algorithm will have found a b.f.s. for the original problem. Phase II is then to proceed with the solution of the original problem, starting from this b.f.s.

Note: the original objective function doesn't enter into Phase I, but it is useful to carry it along as an extra row in the tableau since the algorithm will then arrange for it to be in the appropriate form to start Phase II.

Note also: the Phase I objective must be written in terms of the non-basic variables to begin Phase I. This can also be accomplished in the tableau. We start with

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
y_1	1	1	−1	0	0	1	0	1
y_2	2	−1	0	−1	0	0	1	1
z_3	0	3	0	0	1	0	0	2
Phase II	−6	−3	0	0	0	0	0	0
Phase I	0	0	0	0	0	−1	−1	0

Preliminary step. Add rows 1 and 2 to the Phase I objective so that it is written in terms of non-basic variables.

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
y_1	1	1	−1	0	0	1	0	1
y_2	2	−1	0	−1	0	0	1	1
z_3	0	3	0	0	1	0	0	2
Phase II	−6	−3	0	0	0	0	0	0
Phase I	3	0	−1	−1	0	0	0	2

Begin Phase I.

Pivot on a_{21} to get

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
y_1	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$
x_1	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
z_3	0	3	0	0	1	0	0	2
	0	−6	0	−3	0	0	3	3
	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	0	$-\frac{3}{2}$	$\frac{1}{2}$

Pivot on a_{14} to get

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
z_2	0	3	−2	1	0	2	−1	1
x_1	1	1	−1	0	0	1	0	1
z_3	0	3	0	0	1	0	0	2
	0	3	−6	0	0	6	0	6
	0	0	0	0	0	−1	−1	0

End of Phase I. $y_1 = y_2 = 0$ and we no longer need these variables (so we drop the last two columns and Phase I objective row.) But we do have a b.f.s. to start Phase II with ($x_1 = 1$, $z_2 = 1$, $z_3 = 2$) and the rest of the tableau is already in appropriate form. So we rewrite the last tableau without the y_1, y_2 columns.

Begin Phase II.

	x_1	x_2	z_1	z_2	z_3	
z_2	0	3	-2	1	0	1
x_1	1	1	-1	0	0	1
z_3	0	3	0	0	1	2
	0	3	-6	0	0	6

In one more step we reach the optimum, by pivoting on a_{12}.

	x_1	x_2	z_1	z_2	z_3	
x_2	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
x_1	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$
z_3	0	0	2	-1	1	1
	0	0	-4	-1	0	5

Notice that in fact, the problem we have considered is the same as the problem D, except that x replaces λ and we have added a constraint $2x_2 \leq 3$ (which is not tight at the optimum). It is interesting to compare the final tableau with the final tableau for problem P (shown again below).

In general, artificial variables are needed when there are constraints like

$$\leq -1, \text{ or } \geq 1, \text{ or } = 1,$$

unless constraints happen to be of a special form where it is easy to spot a b.f.s.

If the Phase I objective does not reach zero then the original problem is infeasible.

4.3 Sensitivity: shadow prices

Remarks.

1. Each row of each tableau merely consists of sums of multiples of rows of the original tableau. The objective row = original objective row + scalar multiples of other rows.

   Consider the initial and final tableau for problem P.

initial

1	2	1	0	6
1	-1	0	1	3
1	1	0	0	0

final

0	1	$\frac{1}{3}$	$-\frac{1}{3}$	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	4
0	0	$-\frac{2}{3}$	$-\frac{1}{3}$	-5

In particular, look at the columns 3 and 4, corresponding to variables z_1 and z_2. We can see that

Final row (1) $= \frac{1}{3}$ initial row (1) $-\frac{1}{3}$ initial row (2)
Final row (2) $= \frac{1}{3}$ initial row (1) $+\frac{2}{3}$ initial row (2)
Final objective row = initial objective row $-\frac{2}{3}$ initial row (1) $-\frac{1}{3}$ initial row (2).

In particular, suppose we want to make a small change in b, so we replace $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ by $\begin{pmatrix} 6+\epsilon_1 \\ 3+\epsilon_2 \end{pmatrix}$. Providing ϵ_1, ϵ_2 are small enough they will not affect the sequence of simplex operations. Thus if the constraints move just a little the optimum will still occur with the same variables in the basis. The argument above indicates that the final tableau will be

$$\begin{array}{|cccc|c|}
\hline
0 & 1 & \frac{1}{3} & -\frac{1}{3} & 1+\frac{1}{3}\epsilon_1-\frac{1}{3}\epsilon_2 \\
1 & 0 & \frac{1}{3} & \frac{2}{3} & 4+\frac{1}{3}\epsilon_1+\frac{2}{3}\epsilon_2 \\
\hline
0 & 0 & -\frac{2}{3} & -\frac{1}{3} & -5-\frac{2}{3}\epsilon_1-\frac{1}{3}\epsilon_2 \\
\hline
\end{array}$$

with corresponding solution $x_1 = 4 + \frac{1}{3}\epsilon_1 - \frac{1}{3}\epsilon_2$ and $x_2 = 1 + \frac{1}{3}\epsilon_1 + \frac{2}{3}\epsilon_2$ and objective function value $5 + \frac{2}{3}\epsilon_1 + \frac{1}{3}\epsilon_2$. If ϵ_1, ϵ_2 are such that we have $x_1 < 0$ or $x_2 < 0$ then vertex C is no longer optimal.

The objective function row of the final tableau shows the **sensitivity** of the optimal solution to changes in b and how the optimum value varies with small changes in b. For this reason the values in the objective row are sometimes known as **shadow prices**.

Notice also that being able to see how the final tableau is related to the initial one without looking at the intermediate steps provides a useful way of checking your arithmetic if you suspect you have got something wrong!

2. Notice that for problem P the objective rows at the vertices A, B, C and D are:

$$\begin{array}{ccccc|c}
A & 1 & 1 & 0 & 0 & 0 \\
B & 0 & 2 & 0 & -1 & -3 \\
C & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & -5 \\
D & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -3
\end{array}$$

Compare these values with the basic solutions of the dual problem (on page 7). You will see that the objective row of the simplex tableau corresponding to each b.f.s. of problem P is given by the dual variables for the corresponding basic solution to problem D (after a sign change).

Shadow prices and dual variables are the same thing.

Thus the simplex algorithm can (and should) be viewed as searching amongst primal basic feasible solutions looking for dual feasibility.