# 3 The Simplex Method

## 3.1 Preview of the simplex algorithm

Let us look very closely at problem P and see if we can construct an algorithm that behaves as follows.

**Simplex algorithm**

1. Start with a basic feasible solution.
2. Test — is it optimal?
3. If YES — stop.
4. If NO, move to 'adjacent' and better b.f.s. Return to 2.

We need to pick a b.f.s. to start. Let us take vertex A.

$$x_1 = x_2 = 0;\ z_1 = 6,\ z_2 = 3.$$

[Even for very large problems it is easy to pick a b.f.s. provided the original constraints are $m$ constraints in $n$ variables, $Ax \leq b$ with $b \geq 0$. Once we add slack variables $Ax + z = b$ we have $n + m$ variables and $m$ constraints. If we pick $x = 0$, $z = b$ this is a b.f.s. More about picking the initial b.f. solutions in other cases later.]

Now we can write problem P as:

$$\begin{array}{llcrcl}
x_i, z_i \geq 0 & & & & & \\
 & x_1 + 2x_2 + z_1 & & = & 6 & (1) \\
 & x_1 - x_2 & + z_2 & = & 3 & (2) \\
\max & x_1 + x_2 & & = & f & (0)
\end{array}$$

The peculiar arrangement on the page is deliberate. Now it is obvious that A is not optimal because,

1. At A, $x_1 = x_2 = 0$; $z_1 = 6$, $z_2 = 3$.

2. From the form of the objective function we see that increasing either $x_1$ or $x_2$ will improve the solution.

3. From (1) we see that it is possible to increase $x_1$ to 6 and decrease $z_1$ to 0 without violating this equation or making any variable infeasible.

4. From (2) we see that it is possible to increase $x_1$ to 3 and decrease $z_2$ to 0 before any variable becomes infeasible.

5. Taking (1) and (2) together, then, we can increase $x_1$ to 3, decrease $z_1$ to 3 and decrease $z_2$ to 0, preserving equality in the two constraints, preserving feasibility and increasing the objective function.

That is, we should move to the b.f.s.

$$x_1 = 3,\ x_2 = 0,\ z_1 = 3,\ z_2 = 0,\ (f = 3),$$

which is vertex B. Note that one variable ($x_1$) has entered the basis and one ($z_2$) has left; i.e., we have moved to an 'adjacent' vertex. Why was this easy to see?

1. The objective function $f$ was written in terms of the non-basic variables ($x_1 = x_2 = 0$), so it was easy to see that increasing one of them would improve the solution.

2. Each basic variable ($z_1, z_2$) appeared just once in one constraint, so we could consider the effect of increasing $x_1$ on each basic variable separately when deciding how much we could increase $x_1$.

This suggests we try and write the problem so that the conditions above hold at B, our new b.f.s.. We can do this by adding multiples of the second equation to the others (which is obviously allowed as we are only interested in variables satisfying the constraints.)

So P can be written,

$$\begin{array}{llcrcrcrcll}
x_i, z_i \geq 0 & & & & & & & & & & \\
(1)-(2) & & & 3x_2 & + & z_1 & - & z_2 & = & 3 & (1)' \\
(2) & x_1 & - & x_2 & & & + & z_2 & = & 3 & (2)' \\
(0)-(2) & & & 2x_2 & & & - & z_2 & = & f-3 & (0)'
\end{array}$$

This form of P (equivalent to the original) is what we wanted.

1. The objective function is written in terms of the non-basic variables $x_2, z_2$.

2. Basic variables $x_1, z_1$ each appear just once in one constraint.

The next step is now easy. Remember we are at B:

$$x_1 = 3,\ x_2 = 0,\ z_1 = 3,\ z_2 = 0,\ (f = 3).$$

1. From the objective function it is obvious that increasing $x_2$ is good, whereas increasing $z_2$ would be bad.

2. Equation $(1)'$ shows that we can increase $x_2$ to 1 and decrease $z_1$ to 0.

3. Equation $(2)'$ doesn't impose any restriction on how much we can increase $x_2$ (we just would need to increase $x_1$ also.)

4. Thus we can increase $x_2$ to 1, while decreasing $z_1$ to 0 and increasing $x_1$ to 4.

So we move to vertex C:

$$x_1 = 4, \; x_2 = 1, \; z_1 = 0, \; z_2 = 0, \; (f = 5).$$

Now, rewriting the problem again into the desired form for vertex C we obtain

$$\begin{array}{lcccccl}
x_i, z_i \geq 0 & & & & & & \\
\frac{1}{3}(1)' & & x_2 & +\frac{1}{3}z_1 & -\frac{1}{3}z_2 & = 1 \\
(2)' + \frac{1}{3}(1)' & x_1 & & +\frac{1}{3}z_1 & +\frac{2}{3}z_2 & = 4 \\
(0)' - \frac{2}{3}(1)' & & & -\frac{2}{3}z_1 & -\frac{1}{3}z_2 & = f - 5
\end{array}$$

Now it is clear that we have reached the optimum since

1. We know $x_1 = 4, x_2 = 1, z_1 = 0, z_2 = 0$ is feasible.
2. We know that for any feasible $x$, $z$ we have $f = 5 - \frac{2}{3}z_1 - \frac{1}{3}z_2 \leq 5$. So clearly our solution (with $z_1 = z_2 = 0$) is the best possible.

## 3.2 The simplex algorithm

The procedure just described for problem P can be formalised. Rather than writing out all the equations each time we write just the coefficients in a table known as the **simplex tableau**. To repeat what we have just done, we would write:–

| | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|---|---|---|---|---|---|
| $z_1$ basic | 1 | 2 | 1 | 0 | 6 |
| $z_2$ basic | 1 | $-1$ | 0 | 1 | 3 |
| $a_{0j}$ | 1 | 1 | 0 | 0 | 0 |

If we label the coefficients in the body of the table $(a_{ij})$, the right hand sides of the equations $(a_{i0})$, the coefficients in the expression for the objective function as $(a_{0j})$ and the value of the objective function $-a_{00}$, so the tableau contains

| | |
|---|---|
| $(a_{ij})$ | $a_{i0}$ |
| $a_{0j}$ | $a_{00}$ |

The algorithm is

1. **Choose a variable to enter the basis.** Pick a $j$ such that $a_{0j} > 0$. (The variable corresponding to column $j$ will enter the basis.) If all $a_{0j} \leq 0$, $j \geq 1$, then the current solution is optimal.

   We picked $j = 1$, so $x_1$ is entering the basis.

2. **Find the variable to leave the basis.** Choose $i$ to minimize $a_{i0}/a_{ij}$ from the set $\{i : a_{ij} > 0\}$. If $a_{ij} \leq 0$ for all $i$ then the problem is unbounded (see examples sheet) and the objective function can be increased without limit. If there is more than one $i$ minimizing $a_{i0}/a_{ij}$ the problem has a degenerate basic feasible solution (see example sheet.) For small problems you will be OK if you just choose any one of them and carry on regardless.

   We choose $i = 2$ since $3/1 < 6/1$, so the variable corresponding to equation 2 leaves the basis.

3. **Pivot on the element $a_{ij}$.** (i.e., get the equations into the appropriate form for the new basic solution.)

   (a) multiply row $i$ by $1/a_{ij}$.

   (b) add $-(a_{kj}/a_{ij})\times(\text{row } i)$ to each row $k \neq i$, including the objective function row.

   We obtain as before

| | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|---|---|---|---|---|---|
| $z_1$ basic | 0 | 3 | 1 | $-1$ | 3 |
| $x_1$ basic | 1 | $-1$ | 0 | 1 | 3 |
| $a_{0j}$ | 0 | 2 | 0 | $-1$ | $-3$ |

   which is the appropriate form for the tableau for vertex B.

   Check that repeating these instructions on the new tableau, by pivoting on $a_{12}$, produces the appropriate tableau for vertex C.

| | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|---|---|---|---|---|---|
| $x_2$ basic | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 1 |
| $x_1$ basic | 1 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 4 |
| $a_{0j}$ | 0 | 0 | $-\frac{2}{3}$ | $-\frac{1}{3}$ | $-5$ |

   Note that the columns of the tableau corresponding to the basic variables always make up the columns of an identity matrix.

   Since the bottom row is now all $\leq 0$ the algorithm stops. Don't hesitate to look back at Subsection 3.1 to see why we take these steps.