ADVANCED PROBABILITY **24 lectures**

This course aims to cover the advanced topics at the core of research in probability theory. There is an emphasis on techniques needed for the rigorous analysis of stochastic processes such as Brownian motion.

The measure-theoretic formulation of probability will be reviewed at the beginning of the course, but students unfamiliar with this will need to consolidate their understanding with some further reading, say at the level of Part A of Williams' book.

Review of measure and integration

Measurable functions, integration, expectation of random variables; statements of Fatou's lemma, monotone and dominated convergence theorems; product measures and independence, statement of Fubini's theorem.

Conditional expectation

Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

Martingales

Discrete time martingales, submartingales and supermartingales; optional stopping; Doob's inequalities, upcrossings, convergence theorems, backwards martingales. Applications.

Continuous-time random processes

Kolmogorov's criterion, path regularization theorem for martingales; continuous-time martingales.

Weak convergence in $\mathbb{R}^n$

Definition and characterizations, convergence in distribution, tightness, Prohorov's theorem, characteristic functions, Lévy's continuity theorem.

Sums of independent random variables

Strong law of large numbers, central limit theorem, Cramér's theorem on large deviations.

Brownian motion

Wiener's theorem. Scaling and symmetry properties. Martingales associated to Brownian motion, strong Markov property, reflection principle, hitting times. Sample path properties, recurrence and transience. Brownian motion and the Dirichlet problem. Donsker's invariance principle. Wiener sausage.

Poisson random measures

Definition, compound Poisson processes, Campbell's formula. Infinite divisibility, statement of Lévy-Khinchin theorem. Bartlett's theorem, application to mobile geometric graphs.

Appropriate books

R. Durrett *Probability: Theory and Examples.* Wadsworth 1991
O. Kallenberg *Foundations of Modern Probability.* Springer 1997
J.F.C. Kingman *Poisson Processes.* Oxford Studies in Probability
P. Mörters and Y. Peres *Brownian motion.* Chapters 1,2,3,5 Cambridge University Press 2010
L.C.G. Rogers and D. Williams *Diffusions, Markov processes, and Martingales Vol. I (2nd edition).* Chapters I & II. Wiley 1994
D.W. Stroock *Probability Theory – An analytic view.* Chapters I–V Cambridge University Press 1993
D. Williams *Probability with Martingales.* Cambridge University Press 1991