Example sheet 3

1 Weak convergence

Exercise 1.1. Let $(X_n, n \geq 1)$ be a sequence of independent random variables with uniform distribution on $[0, 1]$. Let $M_n = \max(X_1, \dots, X_n)$. Show that $n(1 - M_n)$ converges in distribution as $n \to \infty$ and determine the limit law.

Exercise 1.2. Let $(X_n, n \geq 0)$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space (M, d).

1. Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$. Show that X_n converges to X_∞ in distribution.

2. Suppose that X_n converges in probability to X_∞. Show that X_n converges in distribution to X_∞.
Hint: use the fact that $(X_n, n \geq 0)$ converges in probability to X_∞ if and only if for every subsequence extracted from $(X_n, n \geq 0)$, there exists a further subsequence converging a.s. to X_∞.

3. If X_n converges in distribution to a constant $X_\infty = c$, then X_n converges in probability to c.

Exercise 1.3. Suppose given sequences $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ of real valued random variables, and two extra random variables X, Y, such that X_n, Y_n respectively converge in distribution to X, Y. Is it true that (X_n, Y_n) converges in distribution to (X, Y)? Show that this is true in the following cases:
1. For every n, the random variables X_n and Y_n are independent, as well as X and Y.

2. Y is a.s. constant (*Hint:* use 3 of the previous question).

Exercise 1.4. Let $d \geq 1$.

1. Show that a finite family of probability measures on $\mathbb{R}^d$ is tight.

2. Assuming Prohorov's theorem for probability measures on $\mathbb{R}^d$, show that if $(\mu_n, n \geq 0)$ is a sequence of non-negative measures on $\mathbb{R}^d$ which is tight and such that

$$\sup_{n \geq 0} \mu_n(\mathbb{R}^d) < \infty,$$

then there exists a subsequence n_k along which μ_n converges weakly to a limit μ.

2 Brownian motion

Exercise 2.1. Show that the standard Brownian motion in $\mathbb{R}^d$ is the unique Gaussian process $(B_t, t \geq 0)$ with $\mathbb{E}[B_t] = 0$ for all $t \geq 0$ and $\mathrm{Cov}(B_s, B_t) = (s \wedge t)I_d$ for every $s, t \geq 0$.

Exercise 2.2. Let B be a standard Brownian in 1 dimension.

(1) Show that a.s.

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

(2) Show that a.s. $B_n/n \to 0$ as $n \to \infty$. Then show that a.s. for n large enough

$$\sup_{t \in [n, n+1]} |B_t - B_n| \leq \sqrt{n}$$

and conclude that $B_t/t \to 0$ as $t \to \infty$ a.s.

(3) Using the time inversion theorem, show that a.s.

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \to \infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

Exercise 2.3. Let B be a standard Brownian motion in 1 dimension. Show that a.s. for all $0 < a < b < \infty$, the Brownian motion B is not monotone on the interval $[a, b]$.

Exercise 2.4. Let $(B_t, t \geq 0)$ be a standard Brownian motion in 1 dimension. Let $T_x = \inf\{t \geq 0 : B_t = x\}$ for $x \in \mathbb{R}$.

1. Prove that T_x has the same distribution as $(x/B_1)^2$ and compute its probability distribution function.

2. For $x, y > 0$, show that

$$\mathbb{P}(T_{-y} < T_x) = \frac{x}{x+y} \quad \text{and} \quad \mathbb{E}[T_{-y} \wedge T_x] = xy.$$

3. Show that if $0 < x < y$, the random variable $T_y - T_x$ has the same law as T_{y-x} and is independent of $\mathcal{F}_{T_x}$ (where $(\mathcal{F}_t, t \geq 0)$ is the natural filtration of Brownian motion).

Hint: the three questions are independent.

Exercise 2.5. Let $(B_t, t \geq 0)$ be a standard Brownian motion in 1 dimension, and let $0 \leq a < b$.

1. Compute the mean and variance of

$$X_n := \sum_{k=1}^{2^n} (B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}})^2.$$

2. Show that X_n converges a.s. and give its limit.

3. Deduce that a.s. there exists no interval $[a, b]$ with $a < b$ such that B is Hölder continuous with exponent $\alpha > 1/2$ on $[a, b]$, i.e. $\sup_{a \le s,t \le b}(|B_t - B_s|/|t-s|^\alpha) < \infty$.

Exercise 2.6. Let $(B_t, t \ge 0)$ be a standard Brownian motion in 1 dimension. Define $G_1 = \sup\{t \le 1 : B_t = 0\}$ and $D_1 = \inf\{t \ge 1 : B_t = 0\}$.

1. Are these random variables stopping times? Show that G_1 has the same distribution as D_1^{-1}.

2. By applying the Markov property at time 1, compute the law of D_1. Deduce that of G_1 (this is called the arcsine law).

Exercise 2.7. Let $(B_t, t \ge 0)$ be a standard Brownian motion in 1 dimension. Define

$$\tau = \inf\{t \ge 0 : B_t = \max_{0 \le s \le 1} B_s\}.$$

Is this a stopping time? *Hint:* First show that $\tau < 1$ a.s.