Example Sheet 2

1. Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave in that

$$\frac{1}{2} \cdot (R(r_1) + R(r_2)) \leq R\left(\frac{1}{2}(r_1 + r_2)\right).$$

2. Without using the commute time identity show that the effective resistance forms a metric on any network with conductances $(c(e))$.

3. Show that if, in a network with source a and sink z, vertices with different voltages are glued together, then the effective resistance from a to z will strictly decrease.

4. Consider simple random walk on the binary tree of depth k with $n = 2^{k+1} - 1$ vertices (the root has degree two and all other nodes except for the leaves have degree 3).

(a) Let a and b be two vertices at level m whose most recent common ancestor c is at level $h < m$. Show that $\mathbb{E}_a[\tau_b] = \mathbb{E}_a[\tau_c] + \mathbb{E}_c[\tau_a]$ and find its value.

(b) Show that the maximal value of $\mathbb{E}_a[\tau_b]$ is achieved when a and b are leaves whose most recent common ancestor is the root of the tree.

5. If τ_y denotes the first hitting time of y, show that

$$\mathbb{E}_x[\tau_y] = \frac{1}{2} \sum_z \deg(z)(R_{\text{eff}}(x, y) + R_{\text{eff}}(y, z) - R_{\text{eff}}(x, z))$$

6. Suppose that Z is a set of states in a Markov chain and that x_0 is a state not in Z. Assume that when the Markov chain is started in x_0, then it visits Z with probability 1. Define the random path $Y_0, Y_1, \ldots$ by $Y_0 := x_0$ and then recursively by letting Y_{n+1} have the distribution of one step of the Markov chain starting from Y_n given that the chain will visit Z before visiting any of $Y_0, Y_1, \ldots, Y_n$ again. However, if $Y_n \in Z$, then the path is stopped and Y_{n+1} is not defined. Show that (Y_n) has the same distribution as loop-erasing a sample of the Markov chain started from x_0 and stopped when it reaches Z. In the case of a random walk, the conditioned path (Y_n) is called the Laplacian random walk from x_0 to Z.

7. Suppose that the graph G has a Hamiltonian path, i.e. there exists a path $(x_k : 1 \leq k \leq n)$ that is a spanning tree. Let X be a simple random walk on G and let $T(A) = \min\{t \geq 0 : X_t \in A\}$ and $T^+(A) = \min\{t \geq 1 : X_t \in A\}$ be the first hitting time and the first return time respectively to the set A. Define

$$q_k = \mathbb{P}_{x_k}\big(T^+(\{x_k\}) > T(\{x_{k+1}, \ldots, x_n\})\big)$$

and show that the number of spanning trees of G equals $\prod_{k<n} q_k \deg(x_k)$.

8. How efficient is Wilsons method? What takes time is to generate a random successor state of a given state. Call this a step of the algorithm. Show that the expected number of steps to generate a random spanning tree rooted at r is

$$\sum_x \frac{\deg(x)}{2|E|}(\mathbb{E}_x[\tau_r] + \mathbb{E}_r[\tau_x]),$$

where $|E|$ is the set of edges and $\deg(x)$ is the degree of the vertex x.

9. Let $G = (V, E)$ be a connected subgraph of the finite connected graph G'. Let T and T' be uniform spanning trees of G and G' respectively. Show that for any edge e of G,

$$\mathbb{P}(e \in T) \geq \mathbb{P}\big(e \in T'\big) .$$

More generally, let B be a subset of E, and show that $\mathbb{P}(B \subseteq T) \geq \mathbb{P}(B \subseteq T')$.

10. Let G be a finite network and $a \neq z$ be two of its vertices. Let i be the unit current flow from a to z. Show that for every edge e, the probability that loop-erased random walk from a to z crosses e minus the probability that it crosses $-e$ is equal to $i(e)$.

11. Let $G = (\mathbb{Z}_n^d, E(\mathbb{Z}_n^d))$ be the d-dimensional torus of side length n, i.e. $\mathbb{Z}_n^d = \{0, \ldots, n-1\}^d$ and $E(\mathbb{Z}_n^d) = \{(x, y) \in \mathbb{Z}_n^d \times \mathbb{Z}_n^d : \|x - y\| = 1\}$. Let $e \in E(\mathbb{Z}_n^d)$. Show that

$$R_{\text{eff}}(e; \mathbb{Z}_n^d) \to \frac{1}{2} \quad \text{as } n \to \infty.$$

12. Let T be the uniform spanning tree in $\mathbb{Z}^2$ and let L be the length of the path in T joining $(0, 0)$ to $(1, 0)$. Show that for all n

$$\mathbb{P}(L \geq n) \geq \frac{1}{8n}.$$

Hint: Consider the event that all the edges on the boundary of the box $[-n, n]^2$ have paths in T joining them with length less than n.

13. Let G be an infinite graph. Let G_n be an exhaustion of G by finite graphs and let μ_n be the UST measure on G_n. The limit of μ_n as $n \to \infty$ exists (same proof as for the wired case) and the limit law is called the free uniform spanning forest. Show that if G is recurrent, then the free uniform spanning forest is the same as the wired uniform spanning forest.

14. Let G be a locally finite graph. Show that all trees of the free and the wired uniform spanning forest are almost surely infinite.

15. Let T be an infinite tree and let $e \in T$. Let $\mathcal{F}$ be the wired uniform spanning forest on T. Show that $\mathbb{P}(e \notin \mathcal{F}) < 1$ if and only if both components of $T \setminus e$ are transient.