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Example Sheet 3

1. Let X be a reversible Markov chain on the finite state space S with invariant distribution π. Let $A \subseteq S$ and set $\tau_A^+ = \min\{t \geq 1 : X_t \in A\}$ to be the first return time to A. Consider now the induced chain on A, i.e. the chain with transition matrix

$$P_A(x,y) = \mathbb{P}_x\left(X_{\tau_A^+} = y\right) \qquad \text{for } x, y \in A.$$

(i) Find the invariant distribution of the chain with matrix P_A and show that it is reversible.

(ii) Let $\varphi : A \to \mathbb{R}$ be a function and define $\widetilde{\varphi}(x) = \mathbb{E}_x\left[\varphi(X_{\tau_A^+})\right]$. Show that

$$P\widetilde{\varphi} = P_A\varphi.$$

(iii) Let γ_A be the spectral gap of the chain with matrix P_A and γ the spectral gap of P. Prove that

$$\gamma_A \geq \gamma.$$

2. Let $A \subseteq [0,n]^2 \cap \mathbb{Z}^2$ obtained by removing the vertices of $[0,n]^2 \cap \mathbb{Z}^2$ with both coordinates even. Consider simple random walk on A and let γ_A be its spectral gap. Using comparison results or otherwise, prove that there exists a positive constant c so that

$$\gamma_A \geq \frac{c}{n^2}.$$

3. Let π be a distribution on the finite set S. Consider the transition matrix $P(x,y) = \pi(y)$ for all $x, y \in S$.

(i) Show that the spectral gap of the matrix P is equal to 1.

(ii) Consider the Markov chain on S^n which at every step chooses one coordinate at random and moves it according to the matrix P. Let γ be its spectral gap. Show that

$$\gamma = \frac{1}{n}.$$

(iii) Deduce the Effron-Stein inequality: let $(X_1, \ldots, X_n)$ and $(X_1', \ldots, X_n')$ be i.i.d. vectors distributed according to $\pi \times \ldots \times \pi$. Let $f : S^n \to \mathbb{R}$ be a function. Show that

$$\mathrm{Var}\left(f(X)\right) \leq \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\Big[\big(f(X_1,\ldots,X_n) - f(X_1,\ldots,X_{i-1},X_i',X_{i+1},\ldots,X_n)\big)^2\Big].$$

4. Show that there exists $\delta \in (0,1)$ sufficiently small such that

$$\sum_{k=1}^{n/2} \binom{n}{\delta k} \frac{\binom{(1+\delta)k}{k}^2}{\binom{n}{k}} \to 0 \qquad \text{as } n \to \infty,$$

thus completing the proof of Theorem 4.15.

5. Consider the exclusion process on the complete graph K_{2n} on $2n$ vertices: there are n indistinguishable black particles and n indistinguishable white particles. Each vertex is occupied by exactly one particle. At each time step, an edge is chosen at random and the particles at its endpoints are swapped. Using path coupling or otherwise, prove an upper bound on the mixing time of order $n \log n$.

6. Let G be a graph of maximum degree Δ. Let $q \geq \Delta + 2$. Show that Glauber dynamics on the space of proper q-colourings is an irreducible Markov chain.

7. Consider the lazy simple random walk on the hypercube $\{0,1\}^n$. Let

$$t_{\mathrm{sep}}(\varepsilon) = \min \left\{ t \geq 0 : \max_{x,y \in \{0,1\}^n} \left(1 - \frac{P^t(x,y)}{\pi(y)} \right) \leq \varepsilon \right\}.$$

Show that for all $\varepsilon \in (0,1)$ we have as $n \to \infty$

$$t_{\mathrm{sep}}(\varepsilon) = n \log n (1 + o(1)),$$

which shows that lazy simple random walk on $\{0,1\}^n$ also exhibits separation cutoff.

8. Let X be a lazy biased walk on $\{0, \ldots, n\}$ with $P(i, i+1) = 1/3 = 1/2 - P(i, i-1)$ for $i \in \{1, \ldots, n-1\}$ and $P(i,i) = 1/2$ and $P(n, n-1) = P(0,1) = 1/2$. Show that X exhibits cutoff at $6n$, i.e. that for all $\varepsilon \in (0,1)$ we have as $n \to \infty$

$$t_{\mathrm{mix}}(\varepsilon) = 6n(1 + o(1)).$$

9. Let G be a finite graph and consider Glauber dynamics for the Ising model on G. Show that for all values of the inverse temperature β we have

$$t_{\mathrm{rel}} \geq \frac{n}{2}.$$

(*Hint:* Use the test function which is the spin at a single vertex.)