Example Sheet 1

1. Let P be the transition matrix of a Markov chain with values in E and let μ and ν be two probability distributions on E. Show that

$$\|\mu P - \nu P\|_{\mathrm{TV}} \leq \|\mu - \nu\|_{\mathrm{TV}} .$$

Deduce that $d(t) = \max_x \left\|P^t(x, \cdot) - \pi\right\|_{\mathrm{TV}}$ is decreasing as a function of t, where π is the invariant distribution.

2. Let $\Omega = \prod_{i=1}^n \Omega_i$, where Ω_i are finite sets. For each i, let μ_i and ν_i be probability distributions on Ω_i and set $\mu = \prod_{i=1}^n \mu_i$ and $\nu = \prod_{i=1}^n \nu_i$. Show that

$$\|\mu - \nu\|_{\mathrm{TV}} \leq \sum_{i=1}^n \|\mu_i - \nu_i\|_{\mathrm{TV}} .$$

3. Let X and Y be Poisson random variables with parameters λ and μ respectively. Writing $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ for their laws, prove that

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{\mathrm{TV}} \leq |\lambda - \mu|.$$

4. Let Y be a random variable with values in $\mathbb{N}$ which satisfies

$$\mathbb{P}(Y = j) \leq c, \text{ for all } j > 0 \text{ and } \mathbb{P}(Y = j) \text{ is decreasing in } j,$$

where c is a positive constant. Let Z be an independent random variable with values in $\mathbb{N}$. Prove that

$$\|\mathbb{P}(Y + Z = \cdot) - \mathbb{P}(Y = \cdot)\|_{\mathrm{TV}} \leq c\mathbb{E}[Z] .$$

5. Let X be a Markov chain and let W and V be random variables taking values in $\mathbb{N}$ and suppose they are independent of X. Prove that

$$\|\mathbb{P}(X_W = \cdot) - \mathbb{P}(X_V = \cdot)\|_{\mathrm{TV}} \leq \|\mathbb{P}(W = \cdot) - \mathbb{P}(V = \cdot)\|_{\mathrm{TV}}$$

6. Let $G = (V, E)$ be a finite connected graph with maximal distance between any two vertices equal to D. Suppose that X is a lazy simple random walk on G. Prove that for all $\varepsilon < 1/2$ we have

$$t_{\mathrm{mix}}(\varepsilon) \geq D/2.$$

7. Let X be a Markov chain in E with transition matrix P and invariant distribution π. Let $A \subseteq E$ be a subset with $\pi(A) \geq 1/8$. Let $\tau_A = \inf\{t \geq 0 : X_t \in A\}$. Prove that there exists a positive constant c so that

$$t_{\mathrm{mix}}(1/4) \geq c \max_x \mathbb{E}_x[\tau_A] .$$

8. Let X be a lazy simple random walk on the d-dimensional discrete torus $\mathbb{Z}_n^d$. Show that there exists a positive constant c (depending on the dimension d) so that

$$t_{\text{mix}}(1/4) \le cn^2.$$

9. A company issues n different coupons. In order to win the prize, a collector needs all n coupons. We suppose that each coupon he acquires is equally likely to be each of the n types. Let X_t denote the number of different types represented among the collector's first t coupons. For $\alpha \in (0,1)$, define $T = \min\{t \ge 0 : X_t = n - n^\alpha\}$.

(a) What is $\mathbb{E}[T]$?

(b) Show that $T/\mathbb{E}[T] \to 1$ in probability as $n \to \infty$.

10. (a) Let S_n be the symmetric group and let $\sigma \in S_n$ be a uniform random permutation. Let X denote the number of fixed points of σ, i.e. the number of $1 \le i \le n$ such that $\sigma(i) = i$. Show that $\mathbb{E}[X] = 1$ and $\text{Var}\,(X) = 1$.

(b) Consider the random transposition shuffle as a method of shuffling a deck of n cards. At each step, the shuffler chooses two cards, L_t and R_t, independently and uniformly at random. If L_t and R_t are different, then transpose them. Otherwise, do nothing. Prove that for any $\varepsilon > 0$ and all n sufficiently large we have

$$t_{\text{mix}}(1/4) \ge \left(\frac{1}{2} - \varepsilon\right) n \log n.$$

11. (a) Let P be a transition matrix. Show that if λ is an eigenvalue, then $|\lambda| \le 1$.

(b) Suppose that P is irreducible and for every x consider the set $T(x) = \{t : P^t(x,x) > 0\}$. Show that $T(x) \subseteq 2\mathbb{Z}$ if and only if -1 is an eigenvalue of P.

12. (a) Let τ be a stopping time for a finite and irreducible Markov chain satisfying $\mathbb{E}[\tau] < \infty$ and $\mathbb{P}_a(X_\tau = a) = 1$. Show that for all x

$$\mathbb{E}_a\left[\sum_{t=0}^{\tau-1} \mathbf{1}(X_t = x)\right] = \pi(x)\mathbb{E}_a[\tau]\,.$$

(b) Consider a finite, irreducible and aperiodic Markov chain. Prove that for all x

$$\pi(x)\mathbb{E}_\pi[\tau_x] = \sum_{t=0}^{\infty} \left(P^t(x,x) - \pi(x)\right).$$

Hint: Count the number of visits to x up until $\tau_x^m = \inf\{t \ge m : X_t = x\}$ in two different ways: using part (a) and also using the convergence to equilibrium theorem.

13. Let P be the transition matrix of a finite reversible chain with invariant distribution π.

Using the Cauchy-Schwarz inequality or otherwise prove that for all x, y and all t

$$\frac{P^{2t}(x,y)}{\pi(y)} \le \sqrt{\frac{P^{2t}(x,x)}{\pi(x)} \cdot \frac{P^{2t}(y,y)}{\pi(y)}} \quad \text{and} \quad P^{2t+2}(x,x) \le P^{2t}(x,x).$$