IB Optimisation: Lecture 12

Mike Tehranchi

University of Cambridge

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The transportation problem.

- m suppliers of a good (for instance, milk) labelled $1, \dots, m$
- supplier i can supply S_i units of the good.
- n destinations (for instance, Sainsbury stores) labelled $1, \dots, n$,
- destination j demanding D_j units.
- For each pair (i, j) there is an associated cost d_{ij} of transport.

Assumption. Total supply equals total demand:

$$\sum_{i=1}^{m} S_i = \sum_{i=1}^{n} D_j$$

The objective to transport as all of the supply of the good to the destination at minimal cost. This is just another linear programming problem:

$$\text{minimise } \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \quad \text{subject to}$$
$$\sum_{j=1}^n x_{ij} = S_i \text{ for all } i = 1, \ldots, m$$
$$\sum_{i=1}^m x_{ij} = D_j \text{ for all } j = 1, \ldots, n$$
$$x_{ij} \geq 0 \text{ for all } i, j.$$

Consider the Lagrangian given by

$$
\begin{aligned}
L(x, \lambda, \mu) &= \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + \sum_{i=1}^{m} \lambda_i \left(S_i - \sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} \mu_j \left(D_j - \sum_{i=1}^{m} x_{ij} \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} (d_{ij} - \lambda_i - \mu_j) x_{ij} + \sum_{i=1}^{m} \lambda_i S_i + \sum_{j=1}^{n} \mu_j D_j .
\end{aligned}
$$

The set of feasible Lagrange multipliers

$$\begin{aligned} \Lambda = &\{(\lambda, \mu) : \inf_{x \geq 0} L(x, \lambda, \mu) > -\infty\} \\ = &\{(\lambda, \mu) : d_{ij} \geq \lambda_i + \mu_j, \text{ for all } i, j\} \end{aligned}$$

If $(x_{ij})_{i,j}$ is optimal for the problem, then there are feasible Lagrange multipliers (λ, μ) satisfying the complementary slackness condition:

$$(d_{ij} - \lambda_i - \mu_j)x_{ij} = 0 \text{ for all } i, j.$$

The transportation algorithm

(0) *Find an initial feasible assignment.* Note that there are $m+n-1$ linear independent relations constraining the feasible set ($\sum_j x_{ij} = S_i$ and $\sum_i x_{ij} = D_j$ but $\sum_i S_i = \sum_j D_j$.) Hence, any b.f.s. will have $m+n-1$ basic variables.

- The North-West method is to put as much flow as feasible in the top left corner. Then move either down or to the right, and put as much as feasible in this cell, and so forth.
- The greedy algorithm is to put as much flow as feasible in the cell with the smallest cost. After adjusting the remaining capacities (which in effect deletes a row or column from the table), put as much flow as feasible into the cell with the second smallest cost, and so forth.

(1) *Assign Lagrange multipliers.* We may take $\lambda_1 = 0$. We enforce complementary slackness by choosing λ_i and μ_j so that $d_{ij} = \lambda_i + \mu_j$ for each basic cell.

(2) *Check for optimality.* The Lagrange multipliers are dual feasible if $d_{ij} \geq \lambda_i + \mu_j$ for each all i, j. If all cells satisfy this inequality, we are done!

(3) *Pivot.* Pick one of the cells (i, j) such that $\lambda_i + \mu_j > d_{ij}$. The rule of thumb is to pick the cell with the largest difference $\lambda_i + \mu_j - d_{ij}$.

Now put an amount $\varepsilon > 0$ units of flow into the pivot cell. At the same time, add or subtract ε from the basic cells to maintain feasibility.

Now choose the largest ε possible such that the flow is feasible.

(4) Go to step (1).

Remark. If the problem is originally posed such that total supply exceeds total demand $\sum_{i=1}^{m} S_i > \sum_{j=1}^{n} D_j$ then we can reformulate the problem by adding another destination with demand $D_{n+1} = \sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j$ and set the unit cost $d_{i,n+1}$ of transport to zero for all suppliers i.

Why does pivoting decrease the total cost?

- For any feasible x and any collection $(\lambda_i)_i$ and $(\mu_j)_j$ of Lagrange multipliers we have

$$
\begin{aligned}
f(x) &= L(x, \lambda, \mu) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} (d_{ij} - \lambda_i - \mu_j) x_{ij} + \sum_{i=1}^{m} \lambda_i S_i + \sum_{j=1}^{n} \mu_j D_j.
\end{aligned}
$$

- Suppose x_0 is the initial feasible assigment.
- We choose $\lambda_{i,0}$ and $\mu_{i,0}$ to enforce complementary slackness: $(d_{ij} - \lambda_{i,0} - \mu_{j,0}) x_{ij,0} = 0$
- Hence $f(x_0) = \sum_{i=1}^{m} \lambda_{i,0} S_i + \sum_{j=1}^{n} \mu_{j,0} D_j$.

- Let $B = \{(i,j) : x_{ij,0} > 0\}$ and $N = \{(i,j) : x_{ij,0} = 0\}$. The set B is the basis for the b.f.s x_0.
- We choose $\lambda_{i,0}$ and $\mu_{i,0}$ such that $\lambda_{i,0} + \mu_{j,0} = d_{ij}$ for all $(i,j) \in B$.
- Suppose the next feasible assignment x_1 is non-degenerate. One of the initial non-basic cells (the pivot cell) is made basic, so that there is a cell $(i_N, j_N) \in N$ such that

$$x_{i_N,j_N,0} = 0 \text{ but } x_{i_N,j_N,1} = \varepsilon > 0.$$

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$$\begin{aligned} f(x_1) = &\sum_{(i,j)\in B} (d_{ij} - \lambda_{i,0} - \mu_{j,0}) x_{ij,1} + \sum_{(i,j)\in N} (d_{ij} - \lambda_{i,0} - \mu_{j,0}) x_{ij,1} \\ &+ f(x_0) \\ =&(d_{i_N,j_N} - \lambda_{i_N,0} - \mu_{j_N,0})\varepsilon + f(x_0). \end{aligned}$$