IB Optimisation: Lecture 10

Mike Tehranchi

University of Cambridge

15 May 2020

Definition

A *network* (or directed graph) is a set $N = \{1, \dots, n\}$ of *nodes* (or vertices), together with a collection $N \times N$ of ordered pairs (i, j) is called *arcs* (or directed edges).

The *flow* in the network is a collection of numbers $(x_{ij})_{i,j}$ indexed by the arcs of the network.

Maximal flow in a network

In this problem, each arc (i,j) is assigned a *capacity* $c_{ij} \geq 0$. (For notational convenience, we will pretend that all nodes are connected by an arc, and set the capacity to zero for non-existent arcs.)

The network has two distinguished nodes, node 1 called the *source* and node n called the *sink*. Flow is conserved for all other nodes. The maximum flow problem is to pump as much flow through the network, from the source to the sink, while respecting the capacity constraints.

Formally,

$$\text{maximise } v \text{ subject to } \quad \begin{array}{l} 0 \le x_{ij} \le c_{ij} \text{ for all } i, j \\ \sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} v & \text{if } i = 1 \\ 0 & \text{if } 1 < i < n \\ -v & \text{if } i = n \end{cases} \end{array}$$

This is a linear program!!

(0) Assign an initial feasible flow $(x_{ij})_{i,j}$ with value v.

(1) Find an *augmenting path*: A collection of nodes $1 = i_1, i_2, \ldots, i_{r-1}, i_r = n$ such that either $c_{i_k,i_{k+1}} - x_{i_k,i_{k+1}} > 0$ or $x_{i_{k+1},i_k} > 0$ for each k.
If no augmenting path exists, then STOP.

(2) If an augmenting path exists, then increase the flow along the path by the amount

$$\delta = \min_{k=1,\ldots,r-1} \max\{c_{i_k i_{k+1}} - x_{i_k i_{k+1}}, x_{i_{k+1} i_k}\}$$

by setting

$$x'_{ij} = \begin{cases} x_{ij} + \delta & \text{if } i,j \text{ are consecutive nodes of path} \\ x_{ij} - \delta & \text{if } j,i \text{ are consecutive nodes of path} \\ x_{ij} & \text{otherwise} \end{cases}$$

Then $(x'_{ij})_{i,j}$ is a feasible flow with value $v' = v + \delta$.

(3) Go to step (1).

We will show that the Ford–Fulkerson algorithm terminates at step (1) if and only if the flow is optimal.

Definition

Let $S \subseteq N$ be a collection of nodes. The complement of S in N is denoted $\bar{S} = N \backslash S$.
A *cut* separating the source = node 1 and the sink = node n is the set of arcs (i, j) with $i \in S$ and $j \in \bar{S}$, where the set S such that $1 \in S$ and $n \in \bar{S}$.

The *capacity* of a cut $(S, \bar{S})$, denoted

$$c(S, \bar{S}) = \sum_{i \in S, j \in \bar{S}} c_{ij},$$

is the sum of all the capacity of the arcs joining S to $\bar{S}$.

Theorem (The max-flow min-cut theorem)

For a network, the maximum value of the flow from node 1 to node n is equal to the minimum of the capacities of all cuts separating 1 and n.

Proof. Fix any feasible flow $(x_{ij})_{i,j\in N}$. For any subset $S, T \subseteq N$ let

$$f(S,T) = \sum_{i\in S, j\in T} x_{ij}$$

be the total flow over arcs joining S and T. Now let $(S, \bar{S})$ be a cut separating 1 and n. Summing the feasibility constraint

$$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = \begin{cases} v & \text{if } i = 1 \\ 0 & \text{if } i = 2, \ldots, n-1 \\ -v & \text{if } i = n \end{cases}$$

over $i \in S$, yields

$$
\begin{aligned}
v &= \sum_{i\in S, j\in N} x_{ij} - \sum_{j\in N, i\in S,} x_{ji} \\
&= f(S,N) - f(N,S) \\
&= [f(S,S) + f(S,\bar{S})] - [f(S,S) + f(\bar{S},S)] \\
&= f(S,\bar{S}) - f(\bar{S},S) \\
&\leq c(S,\bar{S})
\end{aligned}
$$

since for all i, j we have $0 \leq x_{ij} \leq c_{ij}$ and hence $f(S,\bar{S}) \leq c(S,\bar{S})$ and $f(\bar{S},S) \geq 0$. Hence the value of any flow is less than or equal to the capacity of any cut.

To prove that the maximum flow equals the minimum cut, notice that the above inequality becomes an equality if and only if $x_{ij} = c_{ij}$ and $x_{ji} = 0$ for all arcs (i, j) in the cut $(S, \bar{S})$.
So let $(x^*_{ij})_{i,j \in N}$ be the maximal flow. (A maximal flow is guaranteed to exist since the constraint region is compact and the objective function is continuous.) Define a set S recursively as follows
(i) Node 1 is in S.
(ii) If $i \in S$ and $x^*_{ij} < c_{ij}$ then $j \in S$.
(iii) If $i \in S$ and $x^*_{ji} > 0$ then $j \in S$.

Then node n is in $\bar{S}$ since otherwise there would be a path from 1 to n through which more flow could be pumped, contradicting the maximality of $(x_{ij}^*)_{i,j \in N}$. Hence the cut $(S, \bar{S})$ has the property that for every arc $(i,j) \in (S, \bar{S})$ we have $x_{ij}^* = c_{ij}$ and $x_{ji}^* = 0$. □