JRN Lent 2016

Schramm–Loewner evolutions

Examples Sheet 2

1. Denote by $\mathcal{L}$ the set of increasing families of compact $\mathbb{H}$-hulls having the local growth property and parametrized by half-plane capacity. Show that, if $(K_t)_{t\geqslant 0} \in \mathcal{L}$, then for $\lambda \in (0,\infty)$ and $s \geqslant 0$, also the rescaled hulls $(K_t^\lambda)_{t\geqslant 0} \in \mathcal{L}$ and the time-shifted hulls $(K_t^{(s)})_{t\geqslant 0} \in \mathcal{L}$, where

$$K_t^\lambda = \lambda K_{\lambda^{-2}t}, \quad K_t^{(s)} = g_{K_s}(K_{s+t} \setminus K_s) - \xi_s.$$

Express the Loewner transforms $(\xi_t^\lambda)_{t\geqslant 0}$ and $(\xi_t^{(s)})_{t\geqslant 0}$ of $(K_t^\lambda)_{t\geqslant 0}$ and $(K_t^{(s)})_{t\geqslant 0}$ in terms of the Loewner transform $(\xi_t)_{t\geqslant 0}$ of $(K_t)_{t\geqslant 0}$. Justify your answers.

2. Let $\boldsymbol{D} = (D, z_0, z_\infty)$ be a two-pointed (proper, simply-conected) domain. A scale for $\boldsymbol{D}$ is an isomorphism $\boldsymbol{D} \to (\mathbb{H}, 0, \infty)$. Show that there exists a scale σ for $\boldsymbol{D}$, that $\lambda\sigma$ is also a scale for $\boldsymbol{D}$, for all $\lambda \in (0,\infty)$, and that these are all the scales for $\boldsymbol{D}$.

3. Let $\phi : \boldsymbol{D} \to \boldsymbol{D}'$ be an isomorphism of two-pointed domains. Fix scales σ, σ' for $\boldsymbol{D}, \boldsymbol{D}'$. Let $(K_t)_{t\geqslant 0}$ be an SLE(κ) in $\boldsymbol{D}$ of scale σ. Set $K_t' = \phi(K_t)$. Show that, after a suitable rescaling of time, $(K_t')_{t\geqslant 0}$ becomes an SLE(κ) in $\boldsymbol{D}'$ of scale σ'.

4. Let $(K_t)_{t\geqslant 0}$ be an SLE(κ) in $\boldsymbol{D} = (D, z_0, z_\infty)$ of scale σ. Let T be a finite stopping time. Set $D_T = D \setminus K_T$ and $\tilde{K}_t = K_{T+t} \setminus K_T$. Define $z_T \in \delta D_T$ and $\sigma_T : D_T \to \mathbb{H}$ by

$$z_T = g_T^{-1}(\xi_T), \quad \sigma_T(z) = g_T(z) - \xi_T.$$

Show that σ_T is a scale for $\boldsymbol{D}_T = (D_T, z_T, z_\infty)$ and, conditional on $\mathcal{F}_T$, $(\tilde{K}_t)_{t\geqslant 0}$ is an SLE(κ) in $\boldsymbol{D}_T$ of scale σ_T.

5. Let $(X_t(x) : t \in [0, \tau(x)), x \in \mathbb{R} \setminus \{0\})$ be a Bessel flow of parameter $a \in (0,\infty)$. Show that, for all $x, y \in (0,\infty)$ with $x < y$, we have $\tau(x) \leqslant \tau(y)$ and $X_t(x) < X_t(y)$ for all $t < \tau(x)$.

Fix $\lambda \in (0,\infty)$ and set

$$X_t^\lambda(x) = \lambda X_{\lambda^{-2}t}(\lambda^{-1}x), \quad \tau^\lambda(x) = \lambda^2 \tau(\lambda^{-1}x).$$

Show that the distribution of $(X_t^\lambda(x) : t \in [0, \tau^\lambda(x)), x \in \mathbb{R} \setminus \{0\})$ is the same for all λ.

6. Let $(X_t(x) : t \in [0, \zeta(x)), x \in \mathbb{R} \setminus \{0\})$ be a Bessel flow of parameter $1/2$. Show that, almost surely, for all $x \in (0,\infty)$, we have

$$\liminf_{t\to\infty} X_t(x) = 0.$$

7. Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism of initial domains in $\bar{\mathbb{H}}$. Let $(K_t)_{t<T}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property and parametrized

by half-plane capacity, such that $\bar{K}_t \subseteq N \cup I$ for all $t < T$. Consider the conformal isomorphism $\phi_t = g_{\phi(K_t)} \circ \phi \circ g_{K_t}^{-1}$. Show that $(t, z) \mapsto \phi_t(z)$ is differentiable in t and twice differentiable in z in a neighbourhood of (t, ξ_t) for all $t < T$ with

$$\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t).$$

(Carry out the application of l'Hôpital's rule in detail.)

8. Let $\phi : D_0 \to D$ be a conformal isomorphism of planar domains. Show that the map $f \mapsto f \circ \phi^{-1}$ is a linear homeomorphism $\mathcal{D}(D_0) \to \mathcal{D}(D)$.

Let $\rho \in \mathcal{D}(D)$. Show that the image measure of $\rho(x)dx$ by ϕ^{-1} is given by $\rho_0 = (\rho \circ \phi)|\phi'|^2$, and that the map $\rho \mapsto \rho_0$ is a linear homeomorphism $\mathcal{D}(D) \to \mathcal{D}(D_0)$.

For $\gamma_0 \in \mathcal{D}'(D_0)$, define $\gamma \in \mathcal{D}'(D)$ by $\gamma(\rho) = \gamma_0(\rho_0)$. Show that the map $\gamma_0 \mapsto \gamma$ is a linear homeomorphism $\mathcal{D}'(D_0) \to \mathcal{D}'(D)$.

9. Let D be a proper simply connected domain. Show that there is a unique C^∞ function $G_D : \{(z, w) \in D \times D : z \neq w\} \to (0, \infty)$ such that, for all non-negative measurable functions f on D,

$$\mathbb{E}_z \int_0^{T(D)} f(B_t)dt = \int_D G_D(z, w)f(w)dw.$$

Show further that, for any conformal isomorphism $\phi : D_0 \to D$, we have $G_{D_0}(z, w) = G_D(\phi(z), \phi(w))$, and find an explicit form for $G_{\mathbb{D}}$ and $G_{\mathbb{H}}$.

10. Let $\phi : D_0 \to D$ be a conformal isomorphism of proper simply connected domains. Let f be a bounded measurable function on δD_0 and let Γ_0 be a Gaussian free field on D_0 with boundary value f_0. Set $f = f_0 \circ \phi^{-1}$ and set $\Gamma = \Gamma_0 \circ \phi^{-1}$. Show that Γ is a Gaussian free field on D with boundary value f.

11. Let D be a bounded simply connected domain. Recall that there is a complete orthonormal system $(f_n : n \in \mathbb{N})$ in $H_0^1(D)$ and a non-decreasing sequence $(\lambda_n : n \in \mathbb{N})$ in $(0, \infty)$ such that $\sum_n \lambda_n^{-2} < \infty$ and $(-\frac{1}{2}\Delta)f_n = \lambda_n f_n$ for all n. Let Γ be a Gaussian free field on D with zero boundary values. Write $\tilde{\Gamma}$ for the Hilbert space isometry $H^{-1}(D) \to \mathcal{L}^2$ extending Γ. Show that the random variables $Y_n = \lambda_n \tilde{\Gamma}(f_n)$ are independent standard Gaussians. Show further that, almost surely, for all $\rho \in \mathcal{D}(D)$, we have $\Gamma(\rho) = \sum_n \rho_n Y_n$, where $\rho_n = \int_D f_n \rho dx$.