Schramm–Loewner evolutions

Examples Sheet 1

1. Show that a Möbius tranformation f restricts to a conformal automorphism of the upper half plane $\mathbb{H}$ if and only if there exist $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ such that

$$f(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C}.$$

2. Define for $z \in \mathbb{H}$

$$\Psi(z) = \frac{i - z}{i + z}.$$

Show that $\Psi(\mathbb{H}) = \mathbb{D}$ and that $\Psi : \mathbb{H} \to \mathbb{D}$ is a conformal isomorphism.

3. Show that any conformal isomorphism $\phi : D \to D'$ of proper simply connected domains extends uniquely to a homeomorphism of their completions $\hat{D}$ and $\hat{D}'$. Here the completion $\hat{D}$ is taken with respect to a metric on D inherited from the unit disc via a conformal isomorphism.

4. Suppose that σ is a conformal isomorphism $\mathbb{H} \to \mathbb{H}$ fixing 0 and ∞. Show that there exists $\lambda \in (0, \infty)$ such that $\sigma(z) = \lambda z$ for all z.

5. Let D be a proper simply connected domain and let f be a harmonic function on D which extends continuously to $\hat{D}$. Let B be a complex Brownian motion starting from $z \in D$ and set

$$T = \inf\{t \geqslant 0 : B_t \notin D\}.$$

Show that $T < \infty$ and that B_t converges in $\hat{D}$ as $t \uparrow T$, almost surely. Denote the limit by $\hat{B}_T$. Show that $f(\hat{B}_T)$ is integrable and

$$f(z) = \mathbb{E}(f(\hat{B}_T)).$$

6. Use the conformal invariance of harmonic measure to obtain the following formula for the hitting density of Brownian motion on the unit circle, starting from $z \in \mathbb{D}$,

$$h_{\mathbb{D}}(z, t) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad 0 \leqslant t < 2\pi.$$

Find an analogous formula when $|z| > 1$.

7. Let D be a proper simply connected domain and let $x \in D$. Show that there is a unique $r(x) > 0$ and a unique conformal isomorphism $\phi : D \to r(x)\mathbb{D}$ such that $\phi(x) = 0$ and $\phi'(x) = 1$. Show further that, for $y \in D$, as $y \to x$,

$$G_D(x, y) = \frac{1}{\pi} \log \left(\frac{|y - x|}{r(x)} \right) + O(|y - x|).$$

8. Show that there is only one conformal automorphism f of the upper half-plane $\mathbb{H}$ such that $f(z) - z \to 0$ as $z \to \infty$. Show further that for a compact $\mathbb{H}$-hull K, there is at most one conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$. Verify also that, if Φ is defined in a neighbourhood of 0 in $\mathbb{C}$, with

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4)$$

as $z \to 0$, and if $g(z) = -1/\Phi(-1/z) - b$, then, as $z \to \infty$,

$$g(z) = z + \frac{b^2 - c}{z} + O(|z|^{-2}).$$

9. Let K be a compact $\mathbb{H}$-hull and let g_K be its mapping-out function. For $r > 0$ and $x \in \mathbb{R}$, set

$$rK = \{rz : z \in K\}, \quad K + x = \{z + x : z \in K\}.$$

Express the mapping-out functions of rK and $K + x$ in terms of g_K. Hence express the half-plane capacities of rK and $K + x$ in terms of the half-plane capacity a_K of K.

10. Show that $\text{hcap}(K) \leqslant \text{rad}(K)^2$ for all compact $\mathbb{H}$-hulls K. For which K does equality hold?

11. Let K be a compact $\mathbb{H}$-hull and let g_K be its mapping-out function. Suppose that $K \subseteq r\bar{\mathbb{D}} + x$ for some $r > 0$ and $x \in \mathbb{R}$. Show that there is a universal constant $C < \infty$ such that

$$\left| g_K(z) - z - \frac{a_K}{z - x} \right| \leqslant \frac{Cra_K}{|z - x|^2}, \quad |z - x| \geqslant 2r.$$

You may assume the validity of such an estimate in the case where $r = 1$ and $x = 0$.

12. Let $\gamma : [0, \infty) \to \bar{\mathbb{H}}$ be a simple path with $\gamma(0) = 0$, $\gamma_t \in \mathbb{H}$ for all $t > 0$ and $\text{Im}(\gamma_t) \to \infty$ as $t \to \infty$. Set $K_t = \{\gamma_s : 0 < s \leqslant t\}$. Show that $(K_t)_{t \geqslant 0}$ is a strictly increasing family of compact $\mathbb{H}$-hulls, with $\text{hcap}(K_t) \to \infty$ as $t \to \infty$, and having the local growth property.

13. Show that, for the solution flow $(g_t)_{t \geqslant 0}$ of the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z,$$

we have $z(g_t(z) - z) \to 2t$ as $z \to \infty$.

14. Let $T, T' \in (0, \infty]$ and let $\tau : [0, T) \to [0, T')$ be a homeomorphism. Let $(K_t)_{t \in [0,T)}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi_t)_{t \in [0,T)}$. Set $K'_t = K_{\tau(t)}$ and $\xi'_t = \xi_{\tau(t)}$. Show that $(K'_t)_{t \in [0,T')}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi'_t)_{t \in [0,T')}$.

15. Consider the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ generated by the path

$$\gamma_t = \begin{cases} \frac{it}{1-t+it}, & t \leqslant 1, \\ 1 + i(t-1), & t > 1. \end{cases}$$

Does $(K_t)_{t\geqslant 0}$ have the local growth property? Justify your answer.