STOCHASTIC FINANCIAL MODELS

Example Sheet 4

1. Consider the Black–Scholes model with interest rate $r \in \mathbb{R}$ and risky asset $(S_t)_{0\le t\le T}$, having drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Set $X_t = e^{-rT} S_t$. Fix $A \in \mathbb{R}$ and $0 = t_0 \le t_1 \le \cdots \le t_N = T < \infty$ and, for $n = 1, \dots, N$, let θ_n be a bounded $\mathcal{F}_{t_{n-1}}$-measurable random variable. Consider the time-T contingent claim

$$C = e^{rT}\left(A + \sum_{n=1}^{N} \theta_n (X_{t_n} - X_{t_{n-1}})\right).$$

(a) Explain how an investor can replicate C in the market.

(b) Hence show that, for all such claims C, the no-arbitrage time-0 price is given by $e^{-rT}\mathbb{E}^*(C)$ where $\mathbb{P}^*$ is an equivalent probability measure on $\mathcal{F}_T$, to be specified.

2. Show that the Black–Scholes price of a European call option is strictly convex in both the strike price $K > 0$ and the initial stock price $S_0 > 0$, and is decreasing in K and increasing in S_0. Show that the price increases with the interest rate r, and with the expiry $T > 0$ in the case $r \ge 0$.

3. Let $(S_t^0, S_t)_{t\ge 0}$ be a Black–Scholes model with interest rate $r \in \mathbb{R}$, initial stock value $x > 0$, drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Fix $T > 0$ and $K > 0$ consider the European call $Y = (S_T - K)^+$.

(a) Write down the value of Y.

(b) Fix $\alpha > 0$ and set $S_t' = e^{\alpha t} S_t$ and $Y' = (S_T' - K)^+$. Show that $(S_t^0, S_t')_{t\ge 0}$ is also a Black–Scholes model.

(c) What is the value of Y'?

(d) Show that the model $(S_t, S_t')_{t\ge 0}$ contains an arbitrage.

4. Consider the Black–Scholes model with interest rate r, initial stock price x and volatility σ. Show that in this model the Delta and Vega of a European call of maturity T and strike K are given by

$$\Delta = \Phi(d_+), \quad \mathcal{V} = x\phi(d_+)\sqrt{T}$$

where ϕ is the standard normal density function and

$$d_+ = \frac{\log(x/K) + rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

5. Fix times $0 < \tau < T$. A forward start option on stock $(S_t)_{t\ge 0}$ gives the right but not the obligation to buy one unit of the stock at time T for its price at time τ.

(a) Find the price at time 0 of such an option in the Black–Scholes model with interest rate r and volatility σ.

(b) How would this option be hedged?

6. Let $EC(S_0, K, \sigma, r, T)$ denote the Black–Scholes price of a European call option with strike K and expiry T on an asset with initial price S_0 and volatility σ, when the constant interest rate is r. Show that the fair price in the same Black–Scholes model for a down-and-out call with strike K, expiry T and barrier $B < \min\{S_0, K\}$ is given by

$$EC(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2 - 1} EC(B^2/S_0, K, \sigma, r, T).$$

7. A European lookback call option gives the holder the right to buy a unit of stock at time T for its minimum price in the interval $[0, T]$. Thus, for stock price $(S_t)_{0 \le t \le T}$, the pay-off is $S_T - \min_{0 \le t \le T} S_t$. Show that the fair price at time 0 for the lookback call option in the Black-Scholes model of interest rate $r \neq 0$ and volatility σ has the form

$$\frac{S_0 \sigma}{r} \left(\hat{c} \Phi(\hat{c} \sqrt{T}) - c \Phi(c \sqrt{T}) e^{-rT} - \frac{\sigma}{2} \right)$$

where c and $\hat{c}$ are to be determined. Find the fair price also when $r = 0$.