STOCHASTIC FINANCIAL MODELS

Example Sheet 3

1. Consider a multi-period asset price model with numéraire $(S_n^0, S_n)_{0\le n\le T}$. Write X_n for the discounted price S_n/S_n^0. Let $\tilde{\mathbb{P}}$ be an equivalent probability measure such that $\tilde{\mathbb{E}}(|X_n|) < \infty$ for all n. Show that the following are equivalent:

(a) $\tilde{\mathbb{P}}$ is an equivalent martingale measure,

(b) for any previsible self-financing portfolio $(\theta_n^0, \theta_n)_{1\le n\le T}$ with $(\theta_n)_{1\le n\le T}$ bounded, the discounted final value V_T satisfies $\tilde{\mathbb{E}}(V_T) = V_0$.

Hence show that if there exists any equivalent martingale measure then the model has no bounded arbitrage.

2. Let $(S_n)_{0\le n\le T}$ be a binomial model with parameters $a < b$ and interest rate $r \in (a, b)$. Assume that $(1 + a)(1 + b) = 1$ and set $p^* = (r - a)/(b - a)$. Show that the fair price at time 0 for the contingent claim

$$C = \min_{0\le n\le T} S_n$$

is given by

$$V_0 = \sum_{k=0}^{T} p^{*T-k}(1-p^*)^k \sum_{\substack{m=0 \\ m\ge 2k-T}}^{k} \left(\binom{T}{k-m} - \binom{T}{k-m-1} \right) \frac{S_0(1+a)^m}{(1+r)^T}.$$

3. A gambler has the opportunity to bet on a sequence of N coin tosses. The outcomes of these tosses are independent, the nth toss landing heads with probability p_n. Starting from an initial wealth $W_0 = 1$, the gambler may bet any amount in the interval $[0, W_{n-1}]$ on the outcome of the nth toss, where W_{n-1} is his wealth after $n - 1$ tosses. If the gambler calls on the right side, he receives back double his stake, while if he gets it wrong he loses his stake. The gambler wishes to maximize $\mathbb{E}(\log W_N)$. Compute the value function $(V(n, x) : x \in (0, \infty))$ of the gambler, first for $n = N - 1$ and then generally. Hence find the optimal strategy and the maximal value of $\mathbb{E}(\log W_N)$.

4. An investor may choose at the start of day n to invest an amount, x say, in a risky asset. The investor would then receive back a random amount $\xi_n x$ at the end of day n. Any funds which remain uninvested or which are borrowed attract an interest rate of r each day. Assume that the random variables $(\xi_n)_{1\le n\le N}$ are non-negative, integrable, independent and identically distributed. Suppose that the investor has CRRA utility function

$$U(x) = \begin{cases} \dfrac{x^{1-R}}{1-R}, & x \in (0, \infty), \\ -\infty, & \text{otherwise} \end{cases}$$

for some $R \in (0,1)$ and seeks to maximize expected utility at the end of day N. Assume for now that, for some constants $0 < \varepsilon < \lambda < \infty$ we have

$$\xi \in [\varepsilon, \lambda] \quad \text{almost surely.} \tag{1}$$

(a) Consider first the case where the investor chooses to invest each day a proportion $\theta \in [0,1]$ of her current wealth x. Denote by $(V(n,x) : x \in (0,\infty))$ the investor's value function at the end of day n. Find the form of $V(n,.)$, first for $n = N-1$ and then generally.

(b) Show that the investor can achieve a better return by borrowing or shortselling if and only if

$$1 + r < \frac{\mathbb{E}(\xi^{1-R})}{\mathbb{E}(\xi^{-R})} \quad \text{or} \quad \mathbb{E}(\xi) < 1 + r.$$

(c) Comment on the merits of borrowing or shortselling in the absence of condition (1).

5. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $c \in (0,\infty)$ and $s \in [0,\infty)$. Show that the following processes are also Brownian motions: (a) $(-B_t)_{t\geq 0}$, (b) $(c^{-1}B_{c^2t})_{t\geq 0}$, (c) $(B_{s+t} - B_s)_{t\geq 0}$.

6. Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $\theta \in \mathbb{R}$. Define $Q_t = B_t^2 - t$ and $Z_t = e^{\theta B_t - \theta^2 t/2}$. Show directly that the processes $(Q_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are continuous martingales, in a suitable filtration, to be specified.

7. Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $a \geq 0$. Set $T_a = \inf\{t \geq 0 : B_t = a\}$. Use the optional stopping theorem to show that, for all $\lambda \geq 0$,

$$\mathbb{E}\left(e^{-\lambda T_a}\right) = e^{-a\sqrt{2\lambda}}.$$

8. Let $(B_t)_{t\geq 0}$ be a Brownian motion and set

$$M_t = \sup_{0\leq s\leq t} B_s, \quad Z_t = M_t - B_t.$$

(a) Show that (M_t, Z_t) has the same distribution as $\sqrt{t}(M_1, Z_1)$.

(b) Use the reflection principle to find the joint density of (B_1, M_1).

(c) Find the joint density of (M_1, Z_1).

(d) Show that, for $u \neq v$,

$$\mathbb{E}\left(e^{uM_t + vZ_t}\right) = \frac{u\Psi(\sqrt{t}u) - v\Psi(\sqrt{t}v)}{u - v}, \quad \Psi(u) = 2e^{u^2/2}\Phi(u).$$

Suppose now, more generally, that $(B_t)_{t\geq 0}$ is a Brownian motion with drift $c \in \mathbb{R}$. Thus, $B_t = W_t + ct$ for some Brownian motion $(W_t)_{t\geq 0}$.

(e) Show that, for $u \neq -2c$,

$$\mathbb{E}\left(e^{uM_t}\right) = 2\left(\frac{c+u}{2c+u}\right)\Phi((c+u)\sqrt{t})e^{cut+u^2t/2} + 2\left(\frac{c}{2c+u}\right)\Phi(-c\sqrt{t}).$$