STOCHASTIC FINANCIAL MODELS

Example Sheet 2

1. The trees in an orchard are arranged in a rectangular grid. The numbers of apples on each tree are shown in the following array.

2	5	6	1	9	4	3	3	2	9
5	3	8	2	1	4	7	7	1	1
4	9	2	1	4	5	5	7	4	3
1	5	3	3	3	2	4	5	3	7
8	3	4	5	1	2	1	4	1	1
0	2	5	7	8	1	3	1	9	2
3	1	5	6	2	9	4	1	1	1
7	2	3	2	4	5	1	6	5	9
4	3	5	6	1	1	1	2	2	3
8	8	4	5	2	5	7	7	4	2
3	4	2	4	1	9	9	7	1	1

You start at the tree in the leftmost column of the array, at the tree with no apples. You now move one-by-one across the columns, from left to right. You may choose at each step whether to go the tree in the same row or the tree in the row above, or the tree in the row below. Thus at your first step, you can choose to go to a tree with 1 apple, 2 apples or 3 apples; if you choose the tree with 3 apples, then next step you get to choose a tree with 3, 4 or 5 apples. Supposing that you keep all the apples from every tree that you visit in this way, how many apples would you collect if you used the best route? If you were allowed to select the tree that you started at, which one would it be?

2. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0, 1)$, $\mathcal{F}$ is the Borel σ-algebra on $[0, 1)$ and $\mathbb{P}$ is Lebesgue measure. Each $x \in \Omega$ has a binary expansion $x = 0.x_1x_2x_3\ldots$ where $x_n \in \{0, 1\}$ for all n. We make this expansion unique by insisting that, when x is a dyadic rational, we choose the expansion terminating in zeros. Define

$$X(x) = x, \quad X_n(x) = x_n, \quad \mathcal{F}_n = \sigma(X_1, \ldots, X_n).$$

(a) Show that $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables.

(b) Show that $\mathcal{F}_n$ is generated by a finite measurable partition of Ω.

(c) Show that

$$\mathbb{E}(X|\mathcal{F}_n) = \sum_{k=1}^{n} 2^{-k}X_k + 2^{-n-1} \quad \text{almost surely.}$$

3. Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables, having finite mean μ and variance σ^2. Set

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad M_n = S_n - \mu n, \quad Q_n = M_n^2 - \sigma^2 n.$$

Fix $\theta \in \mathbb{R}$ and assume that

$$\psi(\theta) = \log \mathbb{E}(e^{\theta X_1}) < \infty.$$

Set

$$Z_n = e^{\theta S_n - \psi(\theta) n}.$$

Show that the processes $(M_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ are all martingales, with respect to a common filtration, to be specified.

4. Let $(X_n)_{n\geq 0}$ be a simple random walk on the integers starting from 0, with

$$p = \mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) \in (0,1).$$

Fix integers $a, b \geq 1$ and set

$$T = \min\{n \geq 0 : X_n \in \{-a, b\}\}.$$

You may assume throughout that $T < \infty$ almost surely, as you know from Markov Chains. Set

$$M_n = X_n - (2p-1)n, \quad Q_n = M_n^2 - 4p(1-p)n, \quad Z_n = \lambda^{X_n}, \quad \lambda = \frac{1-p}{p}.$$

(a) Show that the processes $(M_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ are all martingales.

(b) Consider the case $p = 1/2$. Use the optional stopping theorem to show that, for all $n \geq 0$,

$$\mathbb{E}(X_{T\wedge n}) = 0, \quad \mathbb{E}(X_{T\wedge n}^2) = \mathbb{E}(T \wedge n).$$

Hence find $\mathbb{P}(X_T = b)$ and $\mathbb{E}(T)$, being careful to justify any limit arguments you use.

(c) Consider the case $p \neq 1/2$. Use the optional stopping theorem to show that

$$\mathbb{P}(X_T = b) = \frac{1 - \lambda^{-a}}{\lambda^b - \lambda^{-a}}$$

and

$$\mathbb{E}(T) = \frac{(1 - \lambda^{-a})b - (\lambda^b - 1)a}{(2p-1)(\lambda^b - \lambda^{-a})}.$$

5. Let $(X_n)_{n\geq 0}$ be a Markov chain with finite state-space E and transition matrix P. Set $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Given a function f on E, we define the function Pf on E by

$$Pf(x) = \sum_{y\in E} p_{xy} f(y).$$

(a) Show that $\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n)$ almost surely.

(b) Show that the following process is a martingale

$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P - I) f(X_k).$$

6. Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables, such that

$$\mathbb{P}(X_1 = 0) < 1, \quad \mathbb{E}(|X_1|) < \infty, \quad \mathbb{E}(X_1) = 0.$$

Set $S_n = X_1 + \cdots + X_n$.

(a) Let $A \in \sigma(S_n)$. Show that $A = \{S_n \in B\}$ for some Borel set B.

(b) Find $\mathbb{E}(S_n|X_1)$ and show that $\mathbb{E}(X_1|S_n) = S_n/n$ almost surely.

(c) Set $M_0 = 0$, $M_1 = S_1$ and $M_2 = S_2$. Use (b) to find random variables $(M_n)_{n\geq 3}$ such that, for all $n \geq 0$,

$$\mathbb{E}(M_{n+1}|M_n) = M_n \quad \text{almost surely}$$

but also such that $(M_n)_{n\geq 0}$ is not a martingale in any filtration.

7. Let Y be a random variable in $\mathbb{R}^d$ and let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$. Assume that $\mathcal{F}_0$ is generated by a countable partition $(B_n : n \in \mathbb{N})$ of Ω. Show that the following are equivalent:

(a) there exists no $\mathcal{F}_0$-measurable random variable Θ in $\mathbb{R}^d$ such that $\Theta.Y \geq 0$ almost surely and $\Theta.Y > 0$ with positive probability,

(b) there exists an equivalent probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_0$ and, almost surely,

$$\tilde{\mathbb{E}}(|Y||\mathcal{F}_0) < \infty, \quad \tilde{\mathbb{E}}(Y|\mathcal{F}_0) = 0.$$

You may assume the equivalence of (a) and (b) in the case $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

8. Consider the single-period asset price model $S_0^0 = S_0 = 1$ and $S_1^0 = 1 + r$, with

$$S_1 = \begin{cases} 1 + a, & \text{with probability } 1 - p, \\ 1 + b, & \text{with probability } p \end{cases}$$

for some $a < b$ and some $p \in (0, 1)$. Show that if $r \leq a$ or $r \geq b$ then this model has an arbitrage.

9. Consider a single-period model with two assets: a riskless bond for which $S_0^0 = 1$ and $S_1^0 = 1 + r$, and a risky asset for which $S_0 = 1$ and S_1 takes three values $a < b < c$ with positive probability. Assume that $a < 1 + r < c$.

(a) Determine all equivalent martingale measures for this model.

(b) Find all functions f such that the contingent claim $C = f(S_1)$ has a replicating portfolio θ.

(c) Show directly that the initial value of the replicating portfolio in (b) is given by $\mathbb{E}(C)/(1 + r)$ for all equivalent martingale measures $\mathbb{P}$.

10. Consider the binomial model obtained as the degenerate case of the model in Example 9 where the risky asset S_1 can no longer take the value b. Assume that the interest rate $r = 0$ and $a < 1 < c$.

(a) Find the optimal investment θ in the risky asset of a utility-maximizing investor with initial wealth w and utility function $U(x) = \sqrt{x}$.

(b) Show that θ is positive if and only if $\mathbb{E}(S_1) > 1$.

11. Consider a single-period model with riskless bond $S_0^0 = S_1^0 = 1$ and risky asset given by $S_0 = 1$ and $S_1 = e^{\sigma Z + \mu}$ where $Z \sim N(0, 1)$ under $\mathbb{P}$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$. Fix $\alpha > 0$ and define $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} \propto e^{\alpha Z}$. Show that $\tilde{\mathbb{P}}$ is an equivalent martingale measure for the model for some α, to be determined.