# Probability and Measure 4

**8.1** Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Let $X$ be a random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{(x-\mu)^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$

Set $Z = (X-\mu)/\sigma$. Show that $Z$ has the standard normal density function. Deduce that $\mathbb{E}(X) = \mu$, $\mathrm{var}(X) = \sigma^2$ and $\phi_X(u) = e^{iu\mu - u^2\sigma^2/2}$. Show that, for $a \neq 0$ and $b \in \mathbb{R}$, the random variable $Y = aX + b$ has a density function of a similar form, for suitable $\mu_Y$ and $\sigma_Y$, to be determined.

**8.2** Let $X = (X_1, \dots, X_n)$ be a Gaussian random variable in $\mathbb{R}^n$ with mean $\mu$ and covariance matrix $V$. Assume that $V$ is invertible and set $Y = (Y_1, \dots, Y_n) = V^{-1/2}(X - \mu)$. Show that $Y_1, \dots, Y_n$ are independent $N(0,1)$ random variables. Show further that we can write $X_2$ in the form $X_2 = aX_1 + Z$ where $Z$ is independent of $X_1$ and determine the distribution of $Z$.

**8.3** Let $X_1, \dots, X_n$ be independent $N(0,1)$ random variables. Show that

$$\left(\overline{X}, \sum_{m=1}^{n} (X_m - \overline{X})^2\right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2\right)$$

have the same distribution, where $\overline{X} = (X_1 + \dots + X_n)/n$.

**9.1** Let $(E, \mathcal{E}, \mu)$ be a measure space and $\theta : E \to E$ a measure-preserving transformation. Show that $\mathcal{E}_\theta := \{A \in \mathcal{E} : \theta^{-1}(A) = A\}$ is a $\sigma$-algebra, and that a measurable function $f$ is $\mathcal{E}_\theta$-measurable if and only if it is *invariant*, that is $f \circ \theta = f$.

**9.2** Show that, if $\theta$ is an ergodic measure-preserving transformation and $f$ is a $\theta$-invariant function, then there exists a constant $c \in \mathbb{R}$ such that $f = c$ a.e..

**9.3** For $x \in [0,1)$, set $\theta(x) = 2x \bmod 1$. Show that $\theta$ is a measure-preserving transformation of $([0,1), \mathcal{B}([0,1)), dx)$, and that $\theta$ is ergodic. Identify the invariant function $\overline{f}$ corresponding to each integrable function $f$.

**9.4** Fix $a \in [0,1)$ and define, for $x \in [0,1)$, $\theta(x) = x + a \bmod 1$. Show that $\theta$ is also a measure-preserving transformation of $([0,1), \mathcal{B}([0,1)), dx)$. Determine for which values of $a$ the transformation $\theta$ is ergodic. *Hint: you may use the fact that any integrable function $f$ on $[0,1)$ whose Fourier coefficients all vanish must itself vanish a.e..* Identify, for all values of $a$, the invariant function $\overline{f}$ corresponding to an integrable function $f$.

**9.5** Call a sequence of random variables $(X_n : n \in \mathbb{N})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *stationary* if for each $n, k \in \mathbb{N}$ the random vectors $(X_1, \ldots, X_n)$ and $(X_{k+1}, \ldots, X_{k+n})$ have the same distribution: for $A_1, \ldots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \ldots, X_{k+n} \in A_n).$$

Show that, if $(X_n : n \in \mathbb{N})$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to X \qquad \text{a.s. and in } L^p,$$

for some random variable $X \in L^p$ and find $\mathbb{E}(X)$.

**10.1** Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $\mathbb{E}(X_n) = \mu$ and $\mathbb{E}(X_n^4) \leq M$ for all $n$, for some constants $\mu \in \mathbb{R}$ and $M < \infty$. Set $P_n = X_1 X_2 + X_2 X_3 + \cdots + X_{n-1} X_n$. Show that $P_n/n$ converges a.s. as $n \to \infty$ and identify the limit.

**10.2** Let $f$ be a bounded continuous function on $(0, \infty)$, having Laplace transform

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx, \quad \lambda \in (0, \infty).$$

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent exponential random variables, of parameter $\lambda$. Show that $\hat{f}$ has derivatives of all orders on $(0, \infty)$ and that, for all $n \in \mathbb{N}$, for some $C(\lambda, n) \neq 0$ independent of $f$, we have

$$(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n) \mathbb{E}\big(f(S_n)\big)$$

where $S_n = X_1 + \cdots + X_n$. Deduce that if $\hat{f} \equiv 0$ then also $f \equiv 0$.

**10.3** For each $n \in \mathbb{N}$, there is a unique probability measure $\mu_n$ on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ such that $\mu_n(A) = \mu_n(UA)$ for all Borel sets $A$ and all orthogonal $n \times n$ matrices $U$. Fix $k \in \mathbb{N}$ and, for $n \geq k$, let $\gamma_n$ denote the probability measure on $\mathbb{R}^k$ which is the law of $\sqrt{n}(x^1, \ldots, x^k)$ under $\mu_n$. Show

(i) if $X \sim N(0, I_n)$ then $X/|X| \sim \mu_n$,

(ii) if $(X_n : n \in \mathbb{N})$ is a sequence of independent $N(0,1)$ random variables and if $R_n = \sqrt{X_1^2 + \cdots + X_n^2}$ then $R_n/\sqrt{n} \to 1$ a.s.,

(iii) for all bounded continuous functions $f$ on $\mathbb{R}^k$, $\gamma_n(f) \to \gamma(f)$, where $\gamma$ is the standard Gaussian distribution on $\mathbb{R}^k$.