Optimization and Control: Examples Sheet 3
Continuous-time models

1. Consider the continuous-time system with scalar state and control variables given by $\dot{x}_t = u_t$, with cost function $Q\int_0^T u_t^2 dt + x_T^2$. Write down and solve the associated HJB equation. Hence show that the minimal cost starting from x is $Qx^2/(Q+T)$ and find an optimal control.

2. Consider the control problem for an inertial particle given by given by $\dot{x}_t = u_t$, $\dot{y}_t = x_t$, with cost function $Q\int_0^T u_t^2 dt + y_T^2$. Re-express this problem in terms of the scalar variable $z_t = y_t + sx_t$, where $s = T - t$. Write down and solve the HJB equation for the new problem. Hence show that the optimal control at time $t = 0$ is given by

$$u_0 = -3Tz_0/(3Q + T^3).$$

3. Miss Prout holds the entire remaining stock of Cambridge elderberry wine for the vintage year 1959. If she releases it at rate u (in continuous time) she realises a unit price $p(u)$. She holds an amount x at time 0 and wishes to release this in such a way as to maximize her total discounted return, $\int_0^\infty e^{-\alpha t} up(u)\, dt$. Consider the particular case $p(u) = u^{-\gamma}$, where the constant γ is positive and less than one. Determine her maximal total discounted return and the optimal initial release rate.

4. Suppose that a particle moves in $\mathbb{R}^d$ with a speed which is a given function $v(x)$ of its position x, but in a direction u which we can choose. The equation of motion is thus $\dot{x}_t = v(x_t)u_t$, where the control u_t is a unit vector. Let $F(x)$ denote the minimal time taken for the particle to reach a set D from a point x outside it. Show that after minimizing over u the optimality equation for F implies that $|\nabla F(x)| = v(x)^{-1}$. (*This is the* eikonal equation *of geometric optics, a short-wavelength form of the wave equation.*) How is the optimal direction at a given point determined from F?

5. Let a be a C^1 function on $\mathbb{R}^d$ with values in the set of $d \times d$ positive-definite matrices. Consider the controllable dynamical system in $\mathbb{R}^d$

$$\dot{x}_t = a(x_t)p_t$$

where the control p_t also takes values in $\mathbb{R}^d$. We seek to minimize, for given $x, y \in \mathbb{R}^d$, the time τ to get from x to y, subject to the constraint

$$p_t^T a(x_t)p_t \leqslant 1, \quad t \geqslant 0.$$

Obtain the following differential equation for the optimal control

$$\dot{p}_t^i = -p_t^T(\partial a/\partial x^i)(x_t)p_t, \quad 0 \leqslant t \leqslant \tau.$$

6. Consider the optimal control problem

$$\text{minimize} \quad \frac{1}{2}\int_0^T u(t)^2 dt \quad \text{subject to} \quad \dot{x}_1 = -x_1 + x_2\,, \quad \dot{x}_2 = -2x_2 + u\,,$$

where u is unrestricted, $x_1(0), x_2(0)$ and T are given, and $x_1(T)$ and $x_2(T)$ are to be made to vanish. By introducing new variables $z_1 = (x_1+x_2)e^t$ and $z_2 = x_2 e^{2t}$, show that the optimal control takes the form $u = \lambda_1 e^t + \lambda_2 e^{2t}$, for some constants λ_1 and λ_2. Find equations for $x_1(0)$, $x_2(0)$ in terms of λ_1, λ_2, and T, and show that these equations determine λ_1 and λ_2 in terms of $x_1(0)$, $x_2(0)$ and T.

Compare the standard linear feedback controller which sets $u(t)$ to $-k_1x_1(t) - k_2x_2(t)$, where k_1 and k_2 are constants. Show that by this method x_1 and x_2 cannot be made to vanish in finite time. Discuss the choice of optimal control with a quadratic performance criterion as opposed to linear feedback control, indicating which is likely to be more appropriate in given circumstances.

7. A princess is jogging with speed r in the counterclockwise direction around a circular running track of radius r, and so has a position whose horizontal and vertical components at time t are $(r\cos t, r\sin t)$, $t \geq 0$. A monster who is initially located at the centre of the circle can move with velocity u_1 in the horizontal direction and u_2 in the vertical direction, where both velocities have a maximum magnitude of 1. The monster wishes to catch the princess in minimal time.

Analyse the monster's problem using Pontryagin's maximum principle. By considering feasible values for the adjoint variables, show that whatever the value of r the monster should always set at least one of $|u_1|$ or $|u_2|$ equal to 1. Show that if $r = \pi/\sqrt{8}$ then the monster catches the princess in minimal time by adopting the uniquely optimal policy $u_1 = 1$, $u_2 = 1$. Is the optimal policy always unique?

[Hint: Let x_1 and x_2 be the differences in the horizontal and vertical directions between the positions of the monster and princess.]

It is not difficult to analyse this problem using ad hoc methods. The point here is to see how easily the maximum principle gives the correct answers without requiring any specially clever argument.

8. In the neoclassical economic growth model, x is the existing capital per worker and u is consumption of capital per worker. The plant equation is

$$\dot{x}_t = f(x_t) - \gamma x_t - u_t,$$

where $f(x)$ is the production per worker, and $-\gamma x$ represents depreciation of capital and change in the size of the workforce. We wish to choose u_t to maximize

$$\int_0^T e^{-\alpha t} g(u_t)\, dt,$$

where $g(u)$ measures utility, is strictly increasing and strictly concave, and T is prescribed.

Show that the optimal control satisfies $g'(u_t) = \lambda_t e^{\alpha t}$ (assuming the maximum is at a stationary point) and

$$\dot{\lambda}_t = (\gamma - f'(x_t))\lambda_t.$$

Hence show that the optimal consumption obeys

$$\dot{u}_t = \frac{1}{\sigma(u_t)}[f'(x_t) - \alpha - \gamma], \qquad \text{where} \qquad \sigma(u) = -\frac{g''(u)}{g'(u)} > 0.$$

(*The function* σ *is called the* elasticity of marginal utility.)

Characterise an equilibrium solution, that is, an $x(0) = \bar{x}$ such that the optimal trajectory is $x(t) = \bar{x}$, $t \geq 0$, and show that this $\bar{x}$ is independent of g.

9. An aircraft flies in straight and level flight at height h, so that lift L balances weight mg. The mass rate of fuel consumption is proportional to the drag, and may be taken as $q = av^2 + b(Lv)^{-2}$, where a and b are constants and v is the speed. Thus

$$\dot{m} = -q = -av^2 - \frac{b}{(mgv)^2}.$$

Show that flying at speed

$$v = \left(\frac{3b}{a(mg)^2}\right)^{1/4}$$

maximizes the total distance the plane can travel. (*You do not need to compute the adjoint path.*)

At what speed should the plane fly in order to maximize the time in the air?

10. In what is known as Zermelo's problem, a straight river has current $c(y)$, running parallel to the banks, where y is the distance from the bank of departure. A boat then crosses the river at constant speed v relative to the water, so that its position (x, y) satisfies $\dot{x} = v\cos\theta + c(y)$, $\dot{y} = v\sin\theta$, where θ is the angle between the direction of the stream and the direction in which the boat is headed. (i) Suppose $c(y) > v$ for all y and the boatman wishes to be carried downstream as little as possible in crossing. Show that he should follow the heading

$$\theta = \cos^{-1}(-v/c(y)).$$

(ii) Suppose the boatman wishes to reach a given point P on the opposite bank in minimal time. Show that he should follow the heading

$$\theta = \cos^{-1}\left(\frac{\lambda_1 v}{1 - \lambda_1 c(y)}\right),$$

where λ_1 is a parameter chosen to make his path pass through the target point.

11. Customers arrive at a queue as a Poisson process of rate λ. They are served at rate $u = u(x)$, where x denotes the current size of the queue. Suppose that cost is incurred as rate $ax + bu$ where a, $b > 0$. The service rate u is regarded as a control variable. The dynamic programming equation in the infinite horizon limit is then

$$\gamma = \inf_u \{ax + bu(x) + \lambda[f(x+1) - f(x)] + u(x)1_{x>0}[f(x-1) - f(x)]\}$$

where γ denotes the average rate at which cost is incurred under the optimal policy and where $f(x)$ denotes the extra cost associated with starting from state x. (Here $1_{x>0} = 0$ if $x = 0$, and $1_{x>0} = 1$ if $x = 1, 2, 3, \ldots$.) Give a brief justification of this equation.

Show that under the constraint that u is a fixed constant, independent of x, and greater that λ then, for some C, there is a solution of the form

$$\gamma = \frac{a\lambda}{u - \lambda} + bu, \quad f(x) = C + \frac{ax(x+1)}{2(u-\lambda)}.$$

i.e., such that $f(x)$ does not grow exponentially in x (which is needed to ensure that $(1/t)Ef(x_t) \to 0$ as $t \to \infty$ and hence, similarly as in the proof for a discrete time model, that γ can be shown to be the time-average cost.)

What is the optimal constant service rate?

Suppose now that we allow u to vary with x, subject to the constraint $m \leq u \leq M$, where $M > \lambda$. What is the policy which results if we carry out one stage of policy improvement to the optimal constant service policy?

12. Consider a problem of movement on the unit interval $0 \leq x \leq 1$ in continuous time, $\dot{x} = u + \epsilon$, where ϵ is white noise of power v. The process terminates at time T when x reaches one end or the other of the the interval. The cost is made up of an integral term $\frac{1}{2}\int_0^T (L + Qu^2)dt$, penalising both control and time spent, and a terminal cost which takes the value C_0 or C_1 according as termination takes place at 0 or 1.

In the deterministic case $v = 0$ it is fairly clear that one will head straight for one of the termination points (which?) at a constant rate (what?). The value function $F(x)$ then has a piecewise linear form, with the possibility of a discontinuity at one of the boundary points if that boundary point is the optimal target from no interior point of the interval.

Show, in the stochastic case, that the dynamic programming equation with the control value optimized out can be linearised by a transformation $F(x) = \alpha \log \phi(x)$ for a suitable constant α, and hence solve the problem.