Optimization and Control: Examples Sheet 2
LQG models

1. A simple model of the rolling motion of a ship represents it as a damped simple pendulum driven by wave motion. For small roll angles the equation is

$$\ddot{\theta} + 2\gamma\omega\dot{\theta} + \omega^2\theta = \omega^2 u,$$

where $\theta(t)$ is the roll angle and $u(t)$ is the effective rolling torque from wave motion; ω and γ are positive constants.

Show that θ and $\dot{\theta}$ can in principle be moved from any values to any other values in an arbitrary short time by an appropriate control u.

2. Consider a scalar deterministic linear system, $x_{k+1} = Ax_k + Bu_k$, with n-horizon total cost function $\sum_{k=0}^{n-1} Qu_k^2 + x_n^2$. By considering a suitable dynamic optimality equation, show that the infimal cost function $V_n(x)$ has the form $\Pi_n x^2$ for some constant Π_n, and that the sequence Π_n^{-1} obeys a linear recurrence. Hence show that

$$\Pi_n = \left(\frac{B^2}{Q(A^2-1)} + \left(1 - \frac{B^2}{Q(A^2-1)}\right) A^{-2n} \right)^{-1}.$$

Under what conditions does Π_n tend to a limit as $n \to \infty$? Investigate the limiting forms of Π_n and of the gain factor Γ_n.

3. Successive attempts are made to regulate the speed of a clock, but each deliberate change in setting introduces also a random change whose size tends to increase with the size of the intended change. Explicitly, let X_n be the error in the speed of the clock after n corrections. On the basis of the observed value of X_n one attempts to correct the speed by an amount U_n. The actual error in speed then becomes

$$X_{n+1} = X_n - U_n + \varepsilon_{n+1}$$

where, conditional on events up to the choice of U_n, the variable ε_{n+1} is normally distributed with zero mean and variance αU_n^2. If, after all attempts at regulation, one leaves the clock with an error x, then there is a cost x^2.

Suppose exactly n attempts are to be made to regulate the clock with initial error x. Determine the optimal policy and the minimal expected cost.

4. Consider the scalar-state control problem with plant equation $X_{k+1} = X_k + U_k + \varepsilon_{k+1}$ and total cost function $\sum_{k=0}^{n-1} U_k^2 + DX_n^2$. Here, the initial state $X_0 = x$ is given, the current state X_k is observable, the horizon point n is prescribed, and the disturbances $\varepsilon_1, \ldots, \varepsilon_n$ are independent with zero mean and common variance v. Find the minimal expected total cost (i) when, for $k = 0, 1, \ldots, n-1$, the control U_k may be chosen to depend on $X_0, \ldots, X_k$ and (ii) when all controls U_k must be chosen at the outset. (These are called *closed loop* and *open loop* controls respectively.)

5. Consider the real-valued system defined by

$$X_{n+1} = aX_n + \xi_n U_n \quad (n = 0, 1, \dots),$$

where U_n is the control at time n and $(\xi_n)_{n\geqslant 0}$ is a sequence of independent random variables with mean b and variance σ^2. Suppose that the cost incurred at time n is $X_n^2 + U_n^2$, and that there are no terminal costs. Find the recursions satisfied by the finite-horizon infimal cost functions. Is the optimal control certainty-equivalent?

6. Suppose that a discrete-time system with d-dimensional state variable x has a plant equation which is linear in state, $x_{k+1} = A_k x_k + b(k, u_k)$, an instantaneous cost $c(k, u_k)$ which is independent of state, and a terminal cost at time n that is a function of $d^T x_n$, for a given vector d. Show that the infimal cost function takes the form $V(k, x) = \phi(k, \xi_k)$, where $\xi_k = d^T z_k$ is the 'predicted miss distance' and $z_k = A_{n-1} \dots A_k x_k$ is the the value that x_n would take starting from x_k at time k if we could take $b = 0$ from then on. Show that the optimal control at time k is also a function of ξ_k and k alone.

7. Consider the linear system given by $X_0 = x, V_0 = v$ and, for $n \geqslant 0$,

$$X_{n+1} = X_n + V_n, \quad V_{n+1} = V_n + U_n + \varepsilon_n.$$

Here (X_n, V_n) are the state variables, representing the position and velocity of a body, U_n is the control variable, which may be chosen as a function of the history of the system up to time n, and $(\varepsilon_n)_{n\geqslant 0}$ is a sequence of independent zero-mean disturbances, with common variance σ^2. The objective is to minimize the expected value of $\sum_{k=0}^{n-1} U_k^2 + P_0 X_n^2$. For a given choice of controls, how does X_n depend on (x, v)? Show that the optimal choice of U_0 is

$$U_0 = -(n-1)P_n(x + nv),$$

where

$$P_n^{-1} = P_0^{-1} + \tfrac{1}{6}n(n-1)(2n-1).$$

8. A one-dimensional model of the problem faced by a juggler trying to balance a light stick with a weight on top is given by the equation

$$\ddot{x}_1 = \alpha(x_1 - u)$$

where x_1 is the horizontal displacement of the top of the stick from some fixed point and u is the horizontal displacement of the bottom. (The stick is assumed to be nearly upright and stationary and $\alpha > 0$ is inversely proportional to the length.) Show that the juggler can control x_1 by manipulating u.

If he tries to balance d such weighted sticks on top of one another, the equations governing stick k ($k = 2, \dots, d$) are (provided the weights on the sticks get smaller fast enough as d increases)

$$\ddot{x}_k = \alpha(x_k - x_{k-1})$$

Show that the d-stick system is fully controllable. [You may find it helpful to take the state vector as $(\dot{x}_1, x_1, \dot{x}_2, x_2, \dots, \dot{x}_d, x_d)^T$.]

9. Consider the following control problem with imperfect state observation

$$Y_n = CX_n + \eta_n, \quad X_{n+1} = AX_n + BU_n + \varepsilon_{n+1}.$$

Here $X_0 \sim N(\mu_0, V_0)$ and $\eta_0, \binom{\varepsilon_1}{\eta_1}, \binom{\varepsilon_2}{\eta_2}, \dots$ are independent non-degenerate zero-mean Gaussians, and U_n is to be chosen as a function of the observations $Y_0, \dots, Y_n$. Compute the conditional distribution of X_0 given Y_0. Find a way to transform this problem to the standard LQG model.

10. Consider the controlled system $X_{n+1} = X_n + U_n + 3\varepsilon_{n+1}$, where $(\varepsilon_n)_{n\geqslant 1}$ is a sequence of independent $N(0,1)$ variables. The instantaneous cost at time n is $X_n^2 + 2U_n^2$. Assuming that X_n is observable at time n, calculate the optimal control under steady-state (stationary) conditions and find the expected cost per unit time incurred when this control is used.

Suppose now that one can observe at time n only $Y_n = X_{n-1} + 2\eta_n$, for $n \geqslant 1$, where $(\eta_n)_{n\geqslant 1}$ is another sequence of independent $N(0,1)$ variables, independent of $(\varepsilon_n)_{n\geqslant 1}$. Show that, in a steady state, the conditional variance of X_n, given $(Y_1, \dots, Y_n)$, is 12.

Determine the optimal control and a recursion for the optimal estimate of state under stationary conditions.