Advanced Probability 2

3.5 Let $(X_n)_{n\geq 0}$ be a Markov chain with state-space E and transition matrix P. Let $f : E \to \mathbb{R}$ be a bounded function. Show that $(f(X_n))_{n\geq 0}$ is a submartingale for all possible initial states $X_0 = x$ if and only if f is subharmonic, that is to say $f \leq Pf$.

3.6 Your winnings per unit stake on game n are ε_n, where $\varepsilon_1, \varepsilon_2, \dots$ are independent random variables with

$$\mathbb{P}(\varepsilon_n = 1) = p, \quad \mathbb{P}(\varepsilon_n = -1) = q,$$

where $p \in (1/2, 1)$ and $q = 1 - p$. Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n - 1$. Your object is to maximize the expected 'interest rate' $\mathbb{E}\log(Z_N/Z_0)$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$. Show that if C is any *previsible* strategy, that is C_n is $\mathcal{F}_{n-1}$-measurable for all n, then $\log Z_n - n\alpha$ is a supermartingale, where

$$\alpha = p\log p + q\log q + \log 2,$$

so that $\mathbb{E}\log(Z_N/Z_0) \leq N\alpha$, but that, for a certain strategy, $\log Z_n - n\alpha$ is a martingale. What is the best strategy?

3.7 Let $(A_n : n \in \mathbb{N})$ be a sequence of independent events all having probability $p > 0$. Fix $N \in \mathbb{N}$ and write T for the first time we achieve a run of N consecutive successes. Thus

$$T = \inf\{n \geq N : 1_{A_n} + \dots + 1_{A_{n-N+1}} = N\}.$$

Show that $\mathbb{P}(T \geq n) \leq C\alpha^n$ for some constants $C < \infty$ and $\alpha \in (0, 1)$ and hence that $\mathbb{E}(T) < \infty$.

3.8 *Wald's identities.* Let $(S_n)_{n\geq 0}$ be a random walk in $\mathbb{R}$, starting from 0, with steps of mean μ and variance $\sigma^2 \in (0, \infty)$. Fix $a, b \in \mathbb{R}$ with $a < 0 < b$ and set

$$T = \inf\{n \geq 0 : S_n \leq a \text{ or } S_n \geq b\}.$$

Show that $\mathbb{E}(T) < \infty$ and $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$. Show further that, in the case $\mu = 0$, we have $\mathbb{E}(S_T^2) = \sigma^2\mathbb{E}(T)$. Show also that, for any $\lambda \in \mathbb{R}$ such that $\mathbb{E}(e^{\lambda S_1}) = 1$, we have $\mathbb{E}(e^{\lambda S_T}) = 1$. In the case of the simple random walk on $\mathbb{Z}$, and for $a, b \in \mathbb{Z}$, use these identities to find $\mathbb{E}(T)$ and $\mathbb{P}(S_T = a)$.

3.9 *Azuma–Hoeffding Inequality.* Let Y be a random variable of mean zero, such that $|Y| \leq c$ for some constant $c < \infty$. Use the convexity of $y \mapsto e^{\theta y}$ on $[-c, c]$ to show that, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta Y}) \leq \cosh\theta c \leq e^{\theta^2 c^2/2}.$$

Now let $(M_n)_{n\geq 0}$ be a martingale, starting from 0, such that $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$, for some constants $c_n < \infty$. Set $v_n = c_1^2 + \dots + c_n^2$. Show that, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta M_n}) \leq e^{\theta^2 v_n/2}.$$

Show further that, for all $x \geq 0$,

$$\mathbb{P}\left(\sup_{k\leq n} M_k \geq x\right) \leq e^{-x^2/(2v_n)}.$$

3.10 Let f be a Lipschitz function on $[0,1]$ of constant K. Thus, for all $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

Denote by f_n the simplest piecewise linear function agreeing with f on $D_n = \{k2^{-n} : k = 0, 1, \dots, 2^n\}$. Then f_n has a derivative f_n' on $[0,1] \setminus D_n$. Set $M_n = f_n' 1_{[0,1]\setminus D_n}$. Show that M_n converges almost everywhere and in L^1 and deduce that there is a bounded Borel function f' on $[0,1]$ such that

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

4.1 Show that the σ-algebra on $C([0,\infty),\mathbb{R})$ generated by the coordinate functions is the same as its Borel σ-algebra for the topology of uniform convergence on compacts.

4.2 Let S and T be stopping times and let $(X_t)_{t\geq 0}$ be a cadlag adapted process, associated to a continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that $S \wedge T$ is a stopping time, that $\mathcal{F}_T$ is a σ-algebra, and that $\mathcal{F}_S \subseteq \mathcal{F}_T$ if $S \leq T$. Show also that $X_T 1_{\{T<\infty\}}$ is an $\mathcal{F}_T$-measurable random variable.

4.3 Let T be an exponential random variable of parameter 1. Set $X_t = e^t 1_{t<T}$. Describe the natural filtration of $(X_t)_{t\geq 0}$. Show that $\mathbb{E}(X_t 1_{\{T>r\}}) = \mathbb{E}(X_s 1_{\{T>r\}})$ for $r \leq s \leq t$, and hence deduce that $(X_t)_{t\geq 0}$ is a cadlag martingale. Determine whether $(X_t)_{t\geq 0}$ is uniformly integrable.

4.4 Let T be a random variable in $[0,\infty)$ having a positive and continuous density function f on $[0,\infty)$. Define the *hazard function* A on $[0,\infty)$ by

$$A(t) = \int_0^t \frac{f(s)ds}{1 - F(s)}$$

where F is the distribution function of T. Show that $A(T)$ is an exponential random variable of parameter 1. Set $X_t = 1_{\{t\geq T\}} - A(T \wedge t)$. Show that $(X_t)_{t\geq 0}$ is a cadlag martingale.

4.5 Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, satisfying the usual conditions, and let $(\xi_t)_{t\geq 0}$ be an adapted integrable process such that $\mathbb{E}(\xi_t|\mathcal{F}_s) = \xi_s$ almost surely, for all $s, t \geq 0$ with $s \leq t$. Show that there is a cadlag martingale $(X_t)_{t\geq 0}$ such that $\xi_t = X_t$ almost surely, for all $t \geq 0$.