

# BOUNDS ON CONNECTIVE CONSTANTS OF REGULAR GRAPHS

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ABSTRACT. The *connective constant*  $\mu$  of a graph  $G$  is the asymptotic growth rate of the number of self-avoiding walks on  $G$  from a given starting vertex. Bounds are proved for the connective constant of an infinite, connected,  $\Delta$ -regular graph  $G$ . The main result is that  $\mu \geq \sqrt{\Delta - 1}$  if  $G$  is vertex-transitive and simple. This inequality is proved subject to weaker conditions under which it is sharp.

## 1. INTRODUCTION

A self-avoiding walk (SAW) is a path on a graph that visits no vertex more than once. SAWs were introduced as a model for long-chain polymers in chemistry (see [9]), and have since been studied intensively by mathematicians and physicists interested in their critical behaviour (see [22]). If the underlying graph has a property of periodicity, the asymptotic behaviour of the number of SAWs of length  $n$  (starting at a given vertex) is exponential in  $n$ , with growth rate called the *connective constant* of the graph. The main purpose of this paper is to explore upper and lower bounds for connective constants.

The principal result of this paper is the following lower bound for the connective constant  $\mu$  of a  $\Delta$ -regular graph. The complementary upper bound  $\mu \leq \Delta - 1$  is very familiar.

**Theorem 1.1.** *Let  $\Delta \geq 2$ , and let  $G$  be an infinite, connected,  $\Delta$ -regular, vertex-transitive, simple graph. Then  $\mu(G) \geq \sqrt{\Delta - 1}$ .*

The problem of counting SAWs is linked in two ways to the study of interacting disordered systems such as percolation and Ising/Potts models. First, the numerical value of  $\mu$  leads to bounds on critical points of such models (see [10, eqns (1.12)–(1.13)] for percolation, and hence Potts models via [11, eqn (5.8)]). Secondly, the SAW problem

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may be phrased in terms of the SAW generating function; this has radius of convergence  $1/\mu$ , and the singularity is believed to have power-law behaviour (for lattice-graphs such as  $\mathbb{Z}^d$ , at least), see [22]. Thus, a lower bound for  $\mu$  may be viewed as an upper bound for the critical point of a certain combinatorial problem.

In Section 2, we introduce notation and definitions used throughout this paper, and in Section 3 we prove a theorem concerning connective constants of general graphs. Inequalities for the connective constant  $\mu(G)$  of a  $\Delta$ -regular graph  $G$  are explored in Section 4, including a re-statement and discussion of Theorem 1.1. It is shown at Theorem 4.2 that a quasi-transitive,  $\Delta$ -regular graph  $G$  satisfies  $\mu(G) = \Delta - 1$  if and only if  $G$  is the  $\Delta$ -regular tree. The proofs of results in Section 4 are found in Section 5.

There are two companion papers, [13, 15]. In [13], we use the Fisher transformation in the context of SAWs on a cubic or partially cubic graph. In particular, we calculate the connective constant of a certain lattice obtained from the hexagonal lattice by applying the Fisher transformation at alternate vertices. In [15], we study strict inequalities between connective constants. It is shown (subject to minor conditions) that  $\mu(G_2) < \mu(G_1)$  if either: (i)  $G_2$  is the quotient graph of  $G_1$  with respect to a non-trivial normal subgroup of an automorphism subgroup that acts transitively, or (ii)  $G_1$  is obtained from  $G_2$  by the addition of a further quasi-transitive family of edges. See also the review [14].

## 2. NOTATION

All graphs in this paper will be assumed infinite, connected, and loopless (a *loop* is an edge both of whose endpoints are the same vertex). In certain circumstances, they are permitted to have *multiple edges* (that is, two or more edges with the same endpoints). A graph  $G = (V, E)$  is called *simple* if it has neither loops nor multiple edges. An edge  $e$  with endpoints  $u, v$  is written  $e = \langle u, v \rangle$ , and two edges with the same endpoints are said to be *parallel*. If  $\langle u, v \rangle \in E$ , we call  $u$  and  $v$  *adjacent* and write  $u \sim v$ . The *degree* of vertex  $v$  is the number of edges incident to  $v$ , denoted  $\deg(v)$ . We assume that the vertex-degrees of a given graph  $G$  are finite with supremum  $\Delta$ , and shall often (but not always) assume  $\Delta < \infty$ . The *graph-distance* between two vertices  $u, v$  is the number of edges in the shortest path from  $u$  to  $v$ , denoted  $d_G(u, v)$ .

A *walk*  $w$  on  $G$  is an alternating sequence  $v_0 e_0 v_1 e_1 \cdots e_{n-1} v_n$  of vertices  $v_i$  and edges  $e_i$  such that  $e_i = \langle v_i, v_{i+1} \rangle$ . We write  $|w| = n$  for the *length* of  $w$ , that is, the number of edges in  $w$ . The walk  $w$  is

called *closed* if  $v_0 = v_n$ . A *cycle* is a closed walk  $w$  with  $v_i \neq v_j$  for  $1 \leq i < j \leq n$ . Thus, two parallel edges form a cycle of length 2.

Let  $n \in \{1, 2, \dots\} \cup \{\infty\}$ . An  $n$ -step self-avoiding walk (SAW) on  $G$  is a walk containing  $n$  edges that includes no vertex more than once. Let  $\sigma_n(v)$  be the number of  $n$ -step SAWs starting at  $v \in V$ . We are interested here in the exponential growth rate of  $\sigma_n(v)$ , and thus we define

$$\underline{\mu}(v) = \liminf_{n \rightarrow \infty} \sigma_n(v)^{1/n}, \quad \bar{\mu}(v) = \limsup_{n \rightarrow \infty} \sigma_n(v)^{1/n}.$$

There is a choice over the most useful way to define the connective constant of an arbitrary graph. One such way is as the constant  $\mu = \mu(G)$  given by

$$(2.1) \quad \mu(G) := \lim_{n \rightarrow \infty} \left( \sup_{v \in V} \sigma_n(v)^{1/n} \right).$$

The limit in (2.1) exists for any graph by the usual argument using subadditivity (see the proof of Theorem 3.2(b)). By Theorem 3.1 below, (2.1) provides an appropriate definition of the connective constant of a quasi-transitive graph. The constant  $\bar{\mu}$  of the forthcoming equation (3.2) provides an alternative definition which is more suitable for a random graph of the type studied by Lacoïn [20].

It will be convenient to consider also SAWs starting at ‘mid-edges’. We identify the edge  $e$  with a point (also denoted  $e$ ) placed at the middle of  $e$ , and we consider walks that start and end at these mid-edges. Such a walk is called *self-avoiding* if it visits no mid-edge or vertex more than once, and its *length* is the number of vertices visited.

The automorphism group of the graph  $G = (V, E)$  is denoted  $\text{Aut}(G)$ . A subgroup  $\mathcal{A} \subseteq \text{Aut}(G)$  is said to *act transitively* on  $G$  if, for  $v, w \in V$ , there exists  $\alpha \in \mathcal{A}$  with  $\alpha v = w$ . It acts *quasi-transitively* if there exists a finite subset  $W \subseteq V$  such that, for  $v \in V$  there exists  $\alpha \in \mathcal{A}$  such that  $\alpha v \in W$ . The graph is called *vertex-transitive* (respectively, *quasi-transitive*) if  $\text{Aut}(G)$  acts transitively (respectively, quasi-transitively).

### 3. BASIC FACTS FOR GENERAL GRAPHS

Hammersley [17] proved that  $\sigma_n(v)^{1/n} \rightarrow \mu$  for a class of graphs including quasi-transitive graphs.

**Theorem 3.1.** [17] *Let  $G = (V, E)$  be an infinite, connected, quasi-transitive graph with finite vertex-degrees. Then*

$$(3.1) \quad \lim_{n \rightarrow \infty} \sigma_n(v)^{1/n} = \mu, \quad v \in V,$$

where  $\mu$  is given in (2.1).

The picture is incomplete in the absence of quasi-transitivity. The following ancillary result provides a partial connection between the  $\bar{\mu}(v)$  and the  $\underline{\mu}(v)$ . The proof appears later in this section.

**Theorem 3.2.** *Let  $G = (V, E)$  be an infinite, connected graph with finite vertex-degrees, and assume there exists  $v \in V$  with  $\bar{\mu}(v) < \infty$ . Let  $\mu$  be given by (2.1).*

- (a) *We have that  $\bar{\mu}(u) = \bar{\mu}(v)$  for all  $u, v \in V$ .*
- (b) *If  $\underline{\mu}(v) = \mu$  for some  $v \in V$ , then  $\underline{\mu}(v) = \mu$  for all  $v \in V$ .*

Part (a) may be found also in [20, Lemma 2.1], which appeared during the writing of this paper. It is in fact only a minor variation of Hammersley’s proof of Theorem 3.1. Subject to the conditions of Theorem 3.2, let  $\bar{\mu} = \bar{\mu}(G)$  be given by

$$(3.2) \quad \bar{\mu} := \bar{\mu}(v), \quad v \in V.$$

By Theorem 3.1,  $\bar{\mu}(G) = \mu(G)$  for any quasi-transitive graph  $G$ . The constant  $\bar{\mu}$  is a more appropriate definition of ‘connective constant’ for a random graph, for which it can be the case that  $\bar{\mu} < \mu$ ; see [20].

Assume that the supremum vertex-degree  $\Delta$  satisfies  $\Delta < \infty$ . It is elementary that

$$(3.3) \quad 1 \leq \underline{\mu}(v) \leq \mu \leq \Delta - 1, \quad v \in V.$$

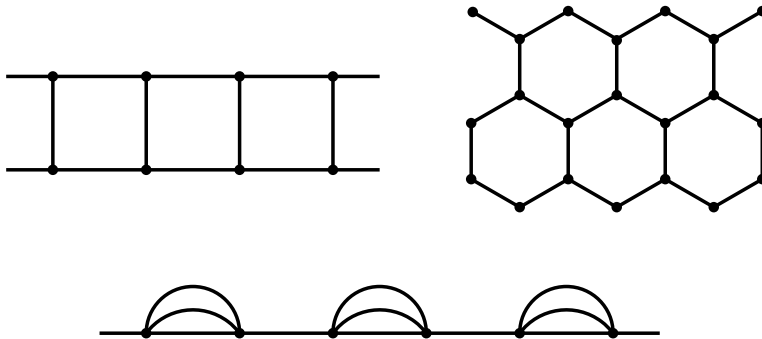


FIGURE 3.1. Three regular graphs: the (doubly-infinite) ladder graph  $\mathbb{L}$ ; the hexagonal tiling  $\mathbb{H}$  of the plane; the bridge graph  $\mathbb{B}_\Delta$  (with  $\Delta = 4$ ) obtained from  $\mathbb{Z}$  by joining every alternating pair of consecutive vertices by  $\Delta - 1$  parallel edges.

The connective constant is known exactly for a limited class of quasi-transitive graphs, of which we mention the ladder  $\mathbb{L}$ , the hexagonal

lattice  $\mathbb{H}$ , and the *bridge graph*  $\mathbb{B}_\Delta$  with degree  $\Delta \geq 2$  of Figure 3.1, for which

$$(3.4) \quad \mu(\mathbb{L}) = \frac{1}{2}(\sqrt{5} + 1), \quad \mu(\mathbb{H}) = \sqrt{2 + \sqrt{2}}, \quad \mu(\mathbb{B}_\Delta) = \sqrt{\Delta - 1}.$$

See [2, p. 184] and [8] for the first two calculations. There is an extensive literature devoted to self-avoiding walks, including numerical upper and lower bounds for connective constants, of which we mention [1, 5, 18, 22].

*Proof of Theorem 3.2.* We adapt and extend an argument of [17]. Let  $u, v$  be neighbours joined by an edge  $e$ . Let  $\pi$  be an  $n$ -step SAW from  $u$ . Either  $\pi$  visits  $v$ , or it does not.

1. If  $\pi$  does not visit  $v$ , we prepend  $e$  to obtain an  $(n + 1)$ -step SAW from  $v$ .
2. If  $\pi$  visits  $v$  after a number  $m < n$  steps, we break  $\pi$  after  $m - 1$  steps, and prepend  $e$  to the first subpath to obtain two SAWs from  $v$ : one of length  $m$  and the other of length  $n - m$ .
3. If  $\pi$  visits  $v$  after  $n$  steps, we remove the final edge and prepend  $e$  to obtain an  $n$ -step SAW from  $v$ .

It follows that

$$(3.5) \quad \sigma_n(u) \leq \sigma_{n+1}(v) + \sum_{m=1}^{n-1} \sigma_m(v)\sigma_{n-m}(v) + \sigma_n(v).$$

Suppose now that  $\bar{\mu}(v) < \infty$ , and let  $\tau > \bar{\mu}(v)$ . There exists  $C = C(\tau) < \infty$  such that

$$(3.6) \quad \sigma_k(v) \leq C\tau^k, \quad k \geq 0.$$

By (3.5),

$$(3.7) \quad \sigma_n(u) \leq C\tau^n(\tau + nC + 1).$$

Hence,  $\bar{\mu}(u) \leq \tau$  and therefore  $\bar{\mu}(u) \leq \bar{\mu}(v)$ . Part (a) follows since  $G$  is connected and undirected.

Turning to part (b), we have in the usual way that

$$(3.8) \quad \sigma_{m+n}(v) \leq \sigma_m(v)\sigma_n,$$

where  $\sigma_n = \sup_{w \in V} \sigma_n(w)$ . Therefore,  $\sigma_{m+n} \leq \sigma_m\sigma_n$ , whence by subadditivity the limit  $\mu$  exists in (2.1). We note in passing that

$$(3.9) \quad \sigma_n \geq \mu^n, \quad n \geq 1.$$

For  $\tau > \mu$ , there exists  $C = C(\tau) < \infty$  such that

$$(3.10) \quad \sigma_n \leq C\tau^n, \quad n \geq 0.$$

Let  $\tau > \mu$ . By (3.8)–(3.10), there exists  $C = C(\tau) < \infty$  such that

$$(3.11) \quad \sigma_{i+j}(u) \leq C\sigma_i(u)\tau^j, \quad u \in V, i, j \geq 0.$$

Set  $n = 2k$  in (3.5), and break the sum into two parts depending on whether or not  $m \leq k$ . By (3.5) and (3.11),

$$(3.12) \quad \sigma_{2k}(u) \leq C\sigma_k(v)(\tau^{k+1} + 2kC\tau^k + \tau^k).$$

Therefore,  $\underline{\mu}(u)^2 \leq \underline{\mu}(v)\tau$ , so that  $\underline{\mu}(u)^2 \leq \underline{\mu}(v)\mu$ . Assume that  $u$  satisfies  $\underline{\mu}(u) = \mu$ . Then  $\underline{\mu}(v) = \mu$ , and the claim follows by iteration.  $\square$

#### 4. CONNECTIVE CONSTANTS OF REGULAR GRAPHS

The graph  $G$  is *regular* (or  $\Delta$ -*regular*) if every vertex has the same degree  $\Delta$ . A 3-regular graph is called *cubic*. In this section, we investigate bounds for the connective constants of infinite regular graphs. The optimal universal lower bound, even restricted to quasi-transitive graphs, is of course the trivial bound  $\mu \geq 1$ . This is achieved when  $\Delta = 3, 4$  by the graphs of Figure 4.1, and by similar constructions for  $\Delta \geq 5$ . Improved bounds may be proved when  $G$  is assumed vertex-transitive.

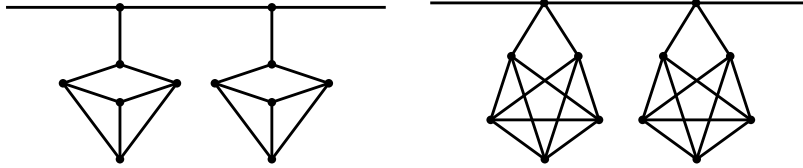


FIGURE 4.1. An infinite line may be decorated in order to obtain regular graphs of degree 3 and 4. Similar constructions yield regular graphs with arbitrary degree  $\Delta$  and connective constant 1.

The main result of this paper, Theorem 1.1, is included in the following theorem, of which the upper bound on  $\mu(G)$  is already well known.

**Theorem 4.1.** *Let  $\Delta \geq 2$ , and let  $G$  be an infinite, connected,  $\Delta$ -regular, vertex-transitive graph. We have  $\mu(G) \leq \Delta - 1$ , and in addition  $\mu(G) \geq \sqrt{\Delta - 1}$  if either*

- (a)  $G$  is simple, or
- (b)  $G$  is non-simple and  $\Delta \leq 4$ .

Part (a) answers a question posed by Itai Benjamini (personal communication). We ask whether the lower bound is *strict* for *simple* graphs, and furthermore, motivated in part by the results of [13], whether or not the golden mean is the sharp lower bound. Secondly, we ask whether part (b) may be extended to larger values of  $\Delta$ . Proofs of theorems in this section are found in Section 5.

For a graph satisfying the initial conditions of Theorem 4.1, we have by Theorem 3.1 that  $\mu(v) = \bar{\mu}(v) = \mu$  for all  $v \in V$ . The Cayley graph (see [3]) of an infinite group with finitely many generators satisfies the hypothesis of Theorem 4.1(a). If the assumption of vertex-transitivity is weakened to quasi-transitivity, the best lower bound is  $\mu \geq 1$ , as illustrated in Figure 4.1.

The upper bound of Theorem 4.1 is an equality for the  $\Delta$ -regular tree  $T_\Delta$ , but is strict for non-trees, even within the larger class of quasi-transitive graphs. We prove the slightly more general fact following, thereby extending an earlier result of Bode [6, Sect. 2.2] for quotients of free groups. (See [19, Lemma 3.3] for a similar result, proved independently of the current work.)

**Theorem 4.2.** *Let  $G = (V, E)$  be an infinite, connected, quasi-transitive graph (possibly with multiple edges), and let  $\Delta \geq 3$ . We have that  $\mu(G) < \Delta - 1$  if either*

- (a)  $G$  is  $\Delta$ -regular and contains a cycle, or
- (b)  $\deg(v) \leq \Delta$  for all  $v \in V$ , and there exists  $w \in V$  with  $\deg(w) \leq \Delta - 1$ .

It is a natural problem to identify operations on graphs under which the connective constant decreases *strictly*. Theorem 4.2 is a step in this direction. When the graphs are required to be regular, this problem may be phrased in terms of graphs and quotient graphs, and a general form of it is formulated and resolved in [15].

We shall deduce Theorem 4.1 from the stronger Theorem 4.3 following. The latter assumes a certain condition which we introduce next, and which plays a role in excluding the graphs of Figure 4.1. The condition is technical, but is satisfied by a variety of graphs of interest.

Let  $G = (V, E)$  be an infinite, connected,  $\Delta$ -regular graph, possibly with multiple edges. For distinct edges  $e, e' \in E$  with a common vertex  $w \in V$ , a SAW is said to *traverse* the triple  $ewe'$  if it contains the mid-edge  $e$  followed consecutively by the vertex  $w$  and the mid-edge  $e'$ . For  $v \in V$ , let  $I(v)$  be the set of infinite SAWs from  $v$ , and  $I(e)$  the corresponding set starting at the mid-edge of  $e \in E$ . Let  $\pi \in I(v)$ , let  $ewe'$  be a triple traversed by  $\pi$ , and write  $\pi_w$  for the finite subwalk of  $\pi$  between  $v$  and  $w$ . Let  $e'' \neq e, e', e''$  be an edge incident to  $w$ . We colour

$e''$  *blue* if there exists  $\pi'' \in I(v)$  that follows  $\pi_w$  to  $w$  and then takes edge  $e''$ , and we colour  $e''$  *red* otherwise. Let  $R_{\pi,w} = \{e_j : j = 1, 2, \dots, r\}$  be the set of red edges corresponding to the pair  $(\pi, w)$ .

We make two notes. First, an edge of the form  $\langle u, w \rangle$  with  $u \in \pi_w$  can be red when seen from  $w$  and blue when seen from  $u$ . Thus, correctly speaking, colour is a property of a *directed* edge. We shall take care over this when necessary. Secondly, suppose there is a group of two or more *parallel* edges  $e'' = \langle w, w' \rangle$  with  $w \in \pi$ ,  $e'' \notin \pi$ . Then all such edges have the same colour. They are all blue if and only if there exists  $\pi'' \in I(v)$  that follows  $\pi_w$  to  $w$  and then takes one of these edges.

The vertex  $v \in V$  is said to *satisfy condition*  $\Pi_v$  if, for all  $\pi \in I(v)$  and all triples  $ewe'$  traversed by  $\pi$ , there exists a set  $F(\pi, w) = \{f_j = \langle x_j, y_j \rangle : j = 1, 2, \dots, |R_{\pi,w}|\}$  of distinct edges of  $G$  such that, for  $1 \leq j \leq |R_{\pi,w}|$ ,

- (a)  $y_j \in \pi_w$ ,  $y_j \neq w$ ,
- (b) there exists a SAW from  $w$  to  $x_j$  with first edge  $e_j$ , that is vertex-disjoint from  $\pi_w$  except at its starting vertex  $w$ .

The graph  $G$  is said to *satisfy condition*  $\Pi$  if every vertex  $v$  satisfies condition  $\Pi_v$ . The set  $F(\pi, w)$  is permitted to contain parallel edges. By reversing the SAWs in (b) above, we see that every edge  $f_j = \langle x_j, y_j \rangle \in F(\pi, w)$  is blue when seen from  $y_j$ . Note that  $F(\pi, v) = \emptyset$  for  $\pi \in I(v)$ .

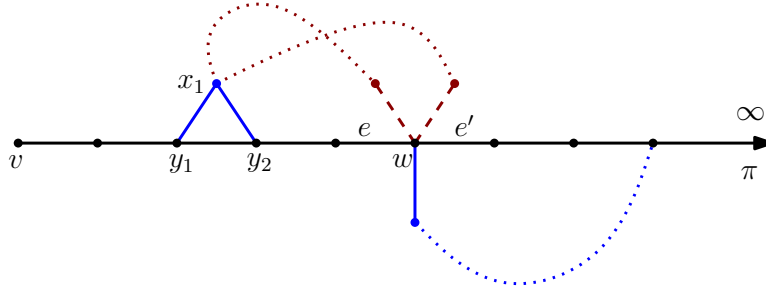


FIGURE 4.2. An illustration of the condition  $\Pi_v$  with  $\Delta = 5$  and  $x_1 = x_2$ . The two edges above  $w$  are red, and the edge below is blue.

Condition  $\Pi_v$  may be expressed in a simpler form for cubic graphs (with  $\Delta = 3$ ). In this case, for each pair  $(\pi, w)$  there exists at most one red edge. Therefore,  $\Pi_v$  is equivalent to the following: for every  $\pi \in I(v)$  and every triple  $ewe'$  traversed by  $\pi$ , there exists  $\pi'' \in I(e'')$  beginning  $e''w''$ , where  $e'' = \langle w, w'' \rangle$  is the third edge incident with  $w$ . It is thus sufficient for a cubic graph  $G$  that every vertex lies in some doubly-infinite self-avoiding walk of  $G$ .



**Theorem 4.3.** *Let  $\Delta \geq 2$ , and let  $G = (V, E)$  be an infinite, connected  $\Delta$ -regular graph. If  $v \in V$  satisfies condition  $\Pi_v$ , we have  $\underline{\mu}(v) \geq \sqrt{\Delta - 1}$ . The bridge graph  $\mathbb{B}_\Delta$  satisfies condition  $\Pi$ , and  $\underline{\mu}(v) = \mu = \sqrt{\Delta - 1}$  for all vertices  $v$ .*

It is trivial that the  $\Delta$ -regular tree  $T_\Delta$  satisfies condition  $\Pi$  and has connective constant  $\Delta - 1$ , and it was noted in (3.3) that  $\Delta - 1$  is an upper bound for connective constants of  $\Delta$ -regular graphs. Let  $\Delta \geq 2$  and  $\sqrt{\Delta - 1} \leq \rho \leq \Delta - 1$ . By replacing the edges of  $T_\Delta$  by finite segments of the bridge graph  $\mathbb{B}_\Delta$ , one may construct graphs satisfying condition  $\Pi$  with connective constant  $\rho$ . Therefore, the set of connective constants of infinite, connected,  $\Delta$ -regular graphs satisfying condition  $\Pi$  is exactly the closed interval  $[\sqrt{\Delta - 1}, \Delta - 1]$ .

## 5. PROOFS OF THEOREMS 4.1–4.3

*Proof of Theorem 4.3.* Let  $G$  satisfy the given conditions. A finite SAW is called *extendable* if it is the starting sequence of some infinite SAW. Let  $v \in V$  satisfy condition  $\Pi_v$ , and let  $\tilde{\sigma}_n$  be the number of extendable  $n$ -step SAWs from  $v$ . We claim that

$$(5.1) \quad \liminf_{n \rightarrow \infty} \tilde{\sigma}_n^{1/n} \geq \sqrt{\Delta - 1},$$

from which the inequality of the theorem follows. The claim is trivial when  $\Delta = 2$ , and we assume henceforth that  $\Delta \geq 3$ . (Since this paper was written, the growth rate for extendable SAWs has been considered in [12].)

Let  $\pi = v_0 e_0 v_1 \cdots e_{2n-1} v_{2n}$  be an extendable  $2n$ -step SAW from  $v_0 = v$ , and, for convenience, augment  $\pi$  with a mid-edge  $e_{-1}$  ( $\neq e_0$ ) incident to  $v_0$ . Thus,  $\pi$  traverses the triples  $e_{s-1} v_s e_s$  for  $0 \leq s < 2n$ . Let  $r_s$  and  $b_s$  be the numbers of red and blue edges, respectively, seen from  $v_s$ , so that

$$(5.2) \quad r_s + b_s = \Delta - 2, \quad 0 \leq s < 2n.$$

We claim that

$$(5.3) \quad \sum_{s=0}^{2n-1} b_s \geq n(\Delta - 2),$$

and the proof of this follows.

For  $0 \leq s < 2n$ , let  $F_s = F(\pi, v_s)$ , and recall that  $F(\pi, v_0) = \emptyset$ . We claim that  $F_s \cap F_t = \emptyset$  for  $s \neq t$ . Suppose on the contrary that  $0 \leq s < t < 2n$  and  $f \in F_s \cap F_t$  for some edge  $f = \langle x, y \rangle$  with  $y = v_u$  and  $u < s$ . See Figure 5.1. There exists a SAW  $\omega_s$  from  $v_s$  to  $x$  such

that: (i) the first edge of  $\omega_s$ , denoted  $e_s$ , is red, and (ii)  $\omega_s$  is vertex-disjoint from  $\pi_{v_s}$  except at  $v_s$ . Similarly, there exists a SAW  $\omega_t$  from  $v_t$  to  $x$  whose first edge  $e_t$  is red, and which is vertex-disjoint from  $\pi_{v_t}$  except at  $v_t$ . Let  $z$  be the earliest vertex of  $\omega_s$  lying in  $\omega_t$ . Consider the infinite SAW  $\omega'$  that starts at  $v_s$ , takes edge  $e_s$ , follows  $\omega_s$  to  $z$ , then  $\omega_t$  and  $e_t$  backwards to  $v_t$ , and then follows  $\pi \setminus \pi_{v_t}$ . Thus,  $\omega'$  is an infinite SAW starting with  $v_s e_s$  that is vertex-disjoint from  $\pi_{v_s}$  except at  $v_s$ . This contradicts the colour of  $e_s$  (seen from  $v_s$ ), and we deduce that  $F_s \cap F_t = \emptyset$  as claimed.

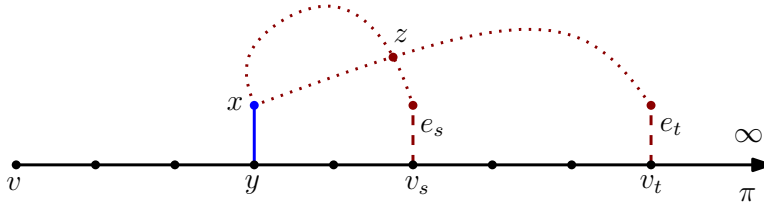


FIGURE 5.1. An illustration of the proof that  $F_s \cap F_t = \emptyset$ .

Since each edge  $f \in F_s$  is blue when seen from one of its endvertices, we have that

$$\sum_{s=0}^{2n-1} b_s \geq \sum_{s=0}^{2n-1} |F_s| = \sum_{s=0}^{2n-1} r_s.$$

The total number of blue/red edges is  $2n(\Delta - 2)$ , and (5.3) follows.

We show next that (5.3) implies the claim of the theorem. A *branch* of  $\pi$  with *root*  $x$  is a (rooted) edge  $e = \langle x, y \rangle$  such that  $x \in \pi$ ,  $x \neq v_{2n}$ ,  $e \notin \pi$ , and the path  $\pi_x$  followed by  $xey$  is an extendable SAW. A set of branches with the same root is called a *fork*. By (5.3),  $\pi$  has at least  $n(\Delta - 2)$  branches, namely the blue edges. Each of these branches gives rise to a further extendable  $2n$ -step SAW from  $v$ , and similarly every such SAW has at least  $n(\Delta - 2)$  such branches. We will show that, regardless of the way in which the branches are grouped into forks, the number of ensuing  $2n$ -step extendable SAWs is at least  $(\Delta - 1)^n$ . In so doing, we will investigate how to group the branches into forks in such a way as to minimize the total number of ensuing  $2n$ -step SAWs.

Let  $n \geq 1$ . The *extendable SAW tree*  $T_n$  at  $v \in V$  is defined as follows. The vertices of  $T_n$  are the extendable  $k$ -step SAWs from  $v$  with  $0 \leq k \leq 2n$ , and two such vertices are joined by an edge if and only if one of the corresponding SAWs is the prolongation by one step of the other. The tree  $T_n$  is rooted at the empty SAW, and it is non-empty since  $G$  is assumed infinite and connected. Let  $L$  be the set of

leaves of  $T_n$  (the root is not considered to be a leaf even if it has degree 1). Thus,  $|L|$  is the number of extendable  $2n$ -step SAWs from  $v$ .

Let  $T$  be a finite rooted tree, and let  $d(w)$  be the down-degree of a vertex  $w$  of  $T$ , that is,  $d(w)$  is the number of edges incident to  $w$  that do not contain the root in the corresponding sub-tree of  $T$ . Let  $\mathcal{P}$  be the set of paths in  $T$  from the root to a leaf. It suffices to prove the next lemma.

**Lemma 5.1.** *Let  $T$  be a finite rooted tree whose down-degrees satisfy  $d(w) \leq \Delta - 1$  and*

$$\sum_{w \in p} (d(w) - 1) \geq n(\Delta - 2), \quad p \in \mathcal{P}.$$

*Then  $T$  has at least  $(\Delta - 1)^n$  leaves.*

*Proof of Lemma 5.1.* As in the above discussion, a *branch* of  $p \in \mathcal{P}$  is an edge of  $T$  with exactly one endvertex in  $p$ . By pruning  $T$ , it suffices to assume that there are exactly  $n(\Delta - 2)$  branches along each  $p \in \mathcal{P}$ . We prove a more general statement by induction on  $B$ , namely the following.

*Claim.* Suppose every  $p \in \mathcal{P}$  has exactly  $B$  branches, where  $B = \alpha(\Delta - 2) + \beta$  with  $\alpha \geq 0$  and  $0 \leq \beta < \Delta - 2$ . Then  $T$  has at least  $g(B)$  leaves where  $g(B) := (\beta + 1)(\Delta - 1)^\alpha$ .

The claim is trivially true when  $B = 1$ , since  $T$  has  $g(2) = 2$  leaves. Suppose  $B_0 \geq 1$  is such that the claim is true for  $B \leq B_0$ , and consider the case  $B = B_0 + 1$ . Let  $B = \alpha(\Delta - 2) + \beta$  as above. Pick  $p \in \mathcal{P}$ , and suppose the fork of branches on  $p$  encountered first when descending from the root has size  $\gamma$  for some  $1 \leq \gamma \leq \Delta - 2$ .

There are two cases depending on whether or not  $\gamma \leq \beta$ . Assume first that  $1 \leq \gamma \leq \beta$ . By the induction hypothesis, the number of leaves is at least  $(\gamma + 1)g(B - \gamma)$ , which satisfies

$$(\gamma + 1)g(B - \gamma) = (\gamma + 1)(\beta - \gamma + 1)(\Delta - 1)^\alpha \geq g(B),$$

as required. In the second case ( $\beta < \gamma \leq \Delta - 2$ ), the corresponding inequality

$$(\gamma + 1)g(B - \gamma) = (\gamma + 1)(\Delta - 2 + \beta - \gamma + 1)(\Delta - 1)^{\alpha-1} \geq g(B)$$

is quickly checked (since the middle expression is an upwards pointing quadratic in  $\gamma$ , it suffices to check the two extremal cases  $\gamma = \beta + 1, \Delta - 2$ ), and the induction is complete.

With  $B \geq n(\Delta - 2)$ , we have  $\alpha \geq n$  and  $g(B) \geq (\Delta - 1)^n$ .  $\square$

By Lemma 5.1 applied to the tree  $T_n$ ,

$$\liminf_{n \rightarrow \infty} \tilde{\sigma}_{2n}^{1/n} \geq \Delta - 1,$$

and (5.1) follows since  $\tilde{\sigma}_k$  is non-decreasing in  $k$ .

Finally, let  $\Delta \geq 2$ . It is easily seen that the bridge graph  $\mathbb{B}_\Delta$  satisfies condition  $\Pi$  and has connective constant  $\sqrt{\Delta - 1}$ , as in (3.4).  $\square$

*Proof of Theorem 4.1.* The upper bound for  $\mu(G)$  is as in (3.3).

(a) Let  $G = (V, E)$  satisfy the given conditions. We claim that, for  $v \in V$ , there exist  $\Delta$  edge-disjoint infinite SAWs from  $v$ . It follows that  $G$  satisfies condition  $\Pi$ , and hence part (a).

The claim is proved as follows. Let  $\lambda_f = \lambda_f(G)$  be the least number of edges whose removal disconnects  $G$  into components at least one of which is finite. By [4, Lemma 3.3] (see also [21, Chap. 12, Prob. 14]), we have that  $\lambda_f = \Delta$ . It is a fairly immediate consequence that there exist  $\lambda_f$  edge-disjoint infinite SAWs from  $v$ . A sketch of this presumably standard fact follows (more elaborate work may be found in [16] and [7, Chap. 8]).

Let  $n \geq 1$ , and let  $B_n$  be the graph obtained from  $G$  by identifying as a single vertex (denoted  $\partial B_n$ ) all vertices having distance  $n + 1$  or more from  $v$ . Since  $v$  has degree  $\Delta$  and  $\lambda_f = \Delta$ , the minimum number of edges whose removal disconnects  $v$  from  $\partial B_n$  is  $\Delta$ . By Menger's theorem (see [7, Sect. 3.3]), there exist  $\Delta$  distinct edge-disjoint SAWs from  $v$  to  $\partial B_n$ . Therefore, for all  $n$ ,  $G$  contains a set of  $\Delta$  edge-disjoint  $n$ -step SAWs from  $v$ . We construct next the required infinite SAWs, and this is where local finiteness is essential.

A set of SAWs is coloured *red* if it comprises  $\Delta$  pairwise edge-disjoint, finite SAWs from  $v$  that, for all large  $n$ , are the initial segments of some set of  $\Delta$  pairwise edge-disjoint,  $n$ -step SAWs. Let  $\eta_1$  be the set of edges incident to  $v$ , considered as the set of 1-step SAWs from  $v$ . By the above and the fact that  $v$  has degree  $\Delta$ ,  $\eta_1$  is red. Let  $\eta_2$  be the set of 2-step SAWs from  $v$ , so that the first steps of  $\eta_2$  comprise the set  $\eta_1$ . Since  $\eta_2$  is finite, it has a  $\Delta$ -subset (denoted  $\eta'_2$ ) that is red. We iterate the last step. Suppose  $m \geq 2$  is such that  $\eta_m, \eta'_m$  have been constructed. Let  $\eta_{m+1}$  be the set of  $(m + 1)$ -step SAWs from  $v$  whose initial  $m$ -step segments comprise  $\eta'_m$ . Since  $\eta_{m+1}$  is finite, it has a  $\Delta$ -subset (denoted  $\eta'_{m+1}$ ) that is red. By induction, there exist  $\Delta$  edge-disjoint infinite SAWs from  $v$ .

(b) When  $\Delta = 2$ ,  $G$  is simple. If  $\Delta = 3$  and  $G$  is non-simple, it is immediate that every  $v$  has property  $\Pi_v$ , and the claim follows by Theorem 4.3. Suppose  $\Delta = 4$ . There are three types of non-simple graph, depending on the groupings of the parallel edges incident to a given vertex. By consideration of these types, we see that only one type merits a detailed argument, namely that in which each vertex

is adjacent to exactly three other vertices, and we restrict ourselves henceforth to this case.

Two paths from  $w \in V$  are called *vertex-disjoint* if  $w$  is their unique common vertex. Let  $\pi \in I(v)$ , with vertex-sequence  $(v, v_1, v_2, \dots)$ . Then  $v_n$  is the endpoint of two vertex-disjoint SAWs of respective lengths  $n$  and  $\infty$ . By vertex-transitivity, for every  $n \geq 1$  and  $w \in V$ ,  $w$  is the endpoint of two vertex-disjoint SAWs of respective lengths  $n$  and  $\infty$ . Since  $G$  is locally finite, every  $w \in V$  is the endpoint of two vertex-disjoint infinite SAWs. We write the last statement as  $v \implies \infty$ .

Let  $\pi \in I(v)$ ,  $w = v_k$  with  $k \geq 1$ , and consider the triple  $ew e'$  traversed by  $\pi$ . By assumption,  $w$  has three neighbours  $w_1, w_2, w_3$  in  $G$ , labelled in such a way that  $w_1 = v_{k-1}$  and  $w_2 = v_{k+1}$ . For some  $i$ , there are two parallel edges of the form  $\langle w, w_i \rangle$ , as illustrated in Figure 5.2.

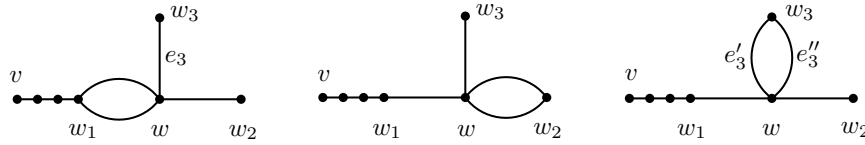


FIGURE 5.2. The three cases in the proof Theorem 4.1(b).

There are several cases to consider. If  $w_3 \in \pi$ , say  $w_3 = v_M$ , any edge  $\langle w, w_3 \rangle$  is red if  $M < k$  and blue otherwise. The situation is more interesting if  $w_3 \notin \pi$ , and we assume this henceforth.

Consider the first case in Figure 5.2 (the second case is similar). The edge  $e_1 = \langle w, w_1 \rangle$  not in  $\pi$  is red (seen from  $w$ ), and contributes itself to the set  $F(\pi, w)$ . Suppose  $e_3 = \langle w, w_3 \rangle$  is red (if it is blue, there is nothing to prove). Since  $w_3 \implies \infty$ , there exists an infinite SAW  $\nu$  from  $w_3$  not using  $e_3$ . Since  $e_3$  is assumed red, there exists a first vertex  $z$  of  $\nu$  lying in  $\pi$ , and furthermore  $z = v_K$  for some  $K < k$ . We add to  $F(\pi, w)$  the last edge of  $\nu$  before  $z$ .

Consider the third case in Figure 5.2, with parallel edges  $e'_3 = \langle w, w_3 \rangle$ ,  $e''_3 = \langle w, w_3 \rangle$ . Since  $e'_3$  and  $e''_3$  have the same colour we may restrict ourselves to the case when both are red. Since  $w_3 \implies \infty$ , there exists an infinite SAW  $\nu$  from  $w_3$  using neither  $e'_3$  nor  $e''_3$ . Since  $e'_3$  and  $e''_3$  are assumed red, there exists an earliest vertex  $v_K$  of  $\nu$  lying in  $\pi$ , with  $K < k$ . Write  $\nu'$  for the sub-path of  $\nu$  that terminates at  $v_K$ , and  $f(\nu')$  for the final edge of  $\nu'$ .

Let  $g = \langle w_3, x \rangle$  be the edge incident to  $w_3$  other than  $e'_3, e''_3$ , and the first edge of  $\nu$ . We construct a path  $\rho$  from  $w_3$  with first edge  $g$ , as follows. Suppose  $\rho$  has been found up to some vertex  $z$ .

1. If  $z$  has been visited earlier by  $\rho$ , we exit  $z$  along the unique edge not previously traversed by  $\rho$ . Such an edge exists since  $G$  is 4-regular.
2. If  $z$  lies in  $\nu'$ , we exit  $z$  along the unique edge lying in neither  $\nu'$  nor the prior part of  $\rho$ .
3. If  $z \in \pi$ , say  $z = v_L$ , we stop the construction, and write  $f(\rho)$  for the final edge traversed.

Since  $e'_3$  and  $e''_3$  are red, Case 3 occurs for some  $L < k$ . By construction,  $\rho$  and  $\nu'$  are edge-disjoint, whence  $f(\rho) \neq f(\nu')$ . Corresponding to the two red edges  $e'_3, e''_3$ , we have the required set  $F(\pi, w) = \{f(\nu'), f(\rho)\}$ .

In conclusion, every  $v \in V$  has property  $\Pi_v$ , and the claim follows by Theorem 4.3.  $\square$

*Proof of Theorem 4.2.* Let  $u \in V$  and let  $e \in E$  be incident to  $u$ . Let  $\sigma_n(u, e)$  be the number of  $n$ -step SAWs from  $u$  that do not traverse  $e$ . We shall prove, subject to either (a) or (b), that there exists  $N = N(G) \geq 3$ , such that

$$(5.4) \quad \sigma_N(u, e) \leq (\Delta - 1)^N - 1 \quad \text{for all such pairs } u, e.$$

Assume first that (a) holds. By quasi-transitivity, there exist  $M, l \in \mathbb{N}$  and a cycle  $\rho$  of length  $l$  such that, for  $v \in V$ , there exist  $w \in V$  and  $\alpha \in \text{Aut}(G)$  such that  $d_G(v, w) < M$  and

$$(5.5) \quad w \in \alpha(\rho).$$

Let  $C(u, e)$  be the subset of  $V$  reachable from  $u$  along paths not using  $e$ . If  $|C(u, e)| < \infty$ , then  $\sigma_n(u, e) = 0$  for large  $n$ , whence (5.4) holds for all  $N \geq N_0$  and some  $N_0 = N_0(u, e)$ . Assume that  $|C(u, e)| = \infty$ . Let  $\pi = (\pi_0, \pi_1, \dots)$  be an infinite SAW from  $u$  not using  $e$ . This walk has a first vertex,  $\pi_R$  say, lying at distance  $4M$  from  $u$ . By the definition of  $M$ , there exists  $k = k(u, e, M)$  satisfying  $R - M \leq k \leq R + M$ , and a  $k$ -step SAW  $\pi'$  from  $u$  not using  $e$ , such that  $\pi'$  has final endpoint  $w'$  lying in  $\alpha'(\rho)$  for some  $\alpha' \in \text{Aut}(G)$ . We may represent the set of SAWs from  $u$ , not using  $e$ , as a subtree of the rooted tree of degree  $\Delta$  (excepting the root, which has degree  $\Delta - 1$ ). By counting the number of paths in that tree, we deduce that, for  $N \geq N_0 := k + l + 1$ , the number of such  $N$ -step walks is no greater than  $(\Delta - 1)^N - 1$ .

Since  $G$  is quasi-transitive,  $N_0 < \infty$  may be picked uniformly in  $u, e$ . Inequality (5.4) is proved in case (a).

If (b) holds, condition (5.5) is replaced by  $\deg(w) \leq \Delta - 1$ , and the conclusion above is valid for  $N \geq k + 2$ . For both cases (a) and (b), (5.4) is proved.

By considering the last edge traversed by a  $(k-1)N$ -step SAW from  $v$ , we have that

$$\sigma_{kN}(v) \leq \sigma_{(k-1)N}(v) [(\Delta - 1)^N - 1], \quad k \geq 2,$$

and, furthermore,  $\sigma_N(v) \leq \Delta [(\Delta - 1)^N - 1]$ . Therefore,

$$\mu = \lim_{n \rightarrow \infty} \sigma_n(v)^{1/n} \leq [(\Delta - 1)^N - 1]^{1/N} < \Delta - 1,$$

and the theorem is proved.  $\square$

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