# NON-SELF-TOUCHING PATHS IN PLANE GRAPHS

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Abstract. A path in a graph $G$ is called non-self-touching if two vertices are neighbours in the path if and only if they are neighbours in the graph. We investigate the existence of doubly infinite non-self-touching paths in infinite plane graphs.

The *matching graph* $G_*$ of an infinite plane graph $G$ is obtained by adding all diagonals to all faces, and it plays an important role in the theory of site percolation on $G$. The main result of this paper is a necessary and sufficient condition on $G$ for the existence of a doubly infinite non-self-touching path in $G_*$ that traverses some diagonal. This is a key step in proving, for quasi-transitive $G$, that the critical points of site percolation on $G$ and $G_*$ satisfy the strict inequality $p_c(G_*) < p_c(G)$, and it complements the earlier result of Grimmett and Li (Random Struct. Alg. 65 (2024) 832–856), proved by different methods, concerning the case of transitive graphs. Furthermore it implies, for quasi-transitive graphs, that $p_u(G) + p_c(G) \geq 1$, with equality if and only if the graph $G_\Delta$, obtained from $G$ by emptying all separating triangles, is a triangulation. Here, $p_u$ is the critical probability for the existence of a unique infinite open cluster.

## 1. Background and main theorem

Some basic facts are presented concerning the existence in an infinite planar graph $G$ of a certain type of doubly infinite path, namely a path $\pi$ with the property that two vertices of $\pi$ are neighbours in $G$ if and only if they are consecutive in $\pi$. Such paths arise naturally in the theory of site percolation.

The graphs considered here are assumed to belong to the set $\mathcal{G}$ of countably infinite, locally finite, 2-connected, simple, plane graphs, embedded in the plane $\mathbb{R}^2$ without accumulation points, and moreover such that all faces have finite diameter. A doubly infinite path $\pi = (\pi_i : -\infty < i < \infty)$ of a graph is called *non-self-touching* if it has the property that $\pi_i \sim \pi_j$ if and only if $|i - j| = 1$. The expression 'doubly infinite non-self-touching path' is abbreviated henceforth to $2\infty$-*nst path.*

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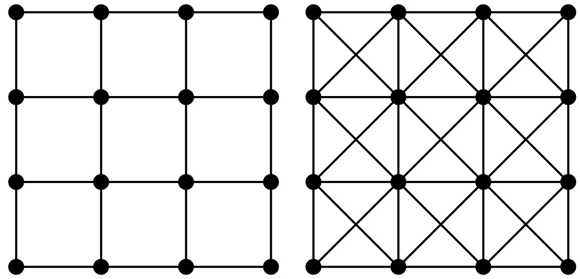

FIGURE 1.1. The square lattice $\mathbb{Z}^2$ and its matching graph.

**Remark 1.1.** *There appears to be no standard expression for the path-property of being non-self-touching, and we adopt this expression for consistency with early work [10]. Possible alternatives include 'chordless path' and 'induced path'. The theory of non-self-touching paths is a matter of potential intrinsic interest in graph theory.*

Which graphs possess a $2\infty$-nst path? We do not have a complete answer to this, but certain cases are described in Section 4.2. For example, every 4-connected $G \in \mathcal{G}$ has a $2\infty$-nst path, and every graph $G \in \mathcal{G}$, embedded in $\mathbb{R}^2$ in such way that its faces have uniformly bounded diameter, has a $2\infty$-nst path.

The *matching graph* $G_*$ of $G \in \mathcal{G}$ is obtained from $G$ by adding all diagonals to all non-triangular faces (see Figure 1.1); the word *diagonal* shall always mean such an edge of $G_*$. The principal purpose of this paper is to prove a property of the pair $(G, G_*)$ of graphs. Evidently, $G_* = G$ if and only if $G$ is a triangulation. Note that, while $G$ is planar, its matching graph $G_*$ is planar if and only if $G$ is a triangulation.

The following graph property is important in the theory of site percolation (see Section 2).

**Definition 1.2.** *The graph $G \in \mathcal{G}$ is said to have property $\Pi$ if $G_*$ has a $2\infty$-nst path that includes some diagonal of some non-triangular face of $G$.*

No triangulation can have property $\Pi$ since a triangulation has no diagonals.

We call a 3-cycle $C$ of a connected plane graph a *separating triangle* if the bounded component of $\mathbb{R}^2 \setminus C$ (termed the *interior* of $C$) intersects one or more edges and/or vertices. If $C$ is a separating triangle of $G \in \mathcal{G}$, then no $2\infty$-nst path may intersect this interior. Thus the interiors of separating triangles may be removed without changing the property of having a $2\infty$-nst path. For $G \in \mathcal{G}$, we write $G_\Delta$ for the subgraph of $G$ obtained by deleting any vertex/edge lying in the interior of any 3-cycle of $G$. We shall normally assume that $G_\Delta \in \mathcal{G}$, thereby eliminating the possibility that $G$ has an infinite nested sequence of 3-cycles. A graph $G \in \mathcal{G}$ is said to be $\triangle$-*empty* if it contains no separating triangle.

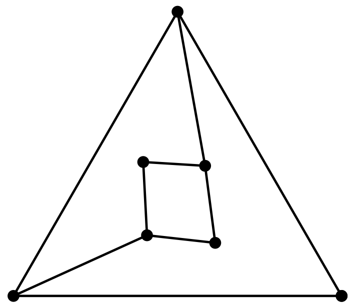

FIGURE 1.2. The graph $G \in \mathcal{G}$ is obtained from the usual triangular lattice by replacing one of more fundamental triangles with a copy of the above. The ensuing graph cannot have property $\Pi$ since no $2\infty$-nst path may penetrate any fundamental triangle.

We prove the straightforward fact (in Theorem 4.2(b)) that a triangulation $T$ has a $2\infty$-nst path if $T_\Delta \in \mathcal{G}$. An example of a graph $G \in \mathcal{G}$ with a separating triangle but without property $\Pi$ is given in Figure 1.2.

Here is the main theorem. Its application to percolation theory is outlined in Section 2.

**Theorem 1.3.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$, and assume $G_\Delta$ is not a triangulation. If $G_*$ has a $2\infty$-nst path, then $G$ has property $\Pi$.*

The basic idea of the proof of Theorem 1.3 is as follows. Since $G_\Delta$ is not a triangulation, it has some face $F$ with four or more edges in its boundary. Assume $G_*$ has a $2\infty$-nst path $\nu$. The target is to show that one can make local changes to $\nu$ in order to obtain a $2\infty$-nst path $\overline{\nu}$ that uses some diagonal of $F$. There are some difficulties in achieving this, and indeed a lesser target is achieved that is sufficient for the theorem. The construction is facilitated by working not with $G$ directly but with the triangulation $\widehat{G}$ (the 'facial graph' of Section 4.3) obtained from $G$ by adding a site to each non-triangular face, and fully connecting this site to the boundary cycle. It is then necessary to understand the relationship between $2\infty$-nst paths of $G_*$ and $2\infty$-nst paths of $\widehat{G}$.

One of the reasons for working with $\widehat{G}$ is that, as a triangulation, one may show the existence of an infinite, nested sequence of cycles with $F$ in their common interior. This permits an iterative approach to the construction of $\overline{\nu}$.

Here is a summary of the contents of this article. The application of Theorem 1.3 to percolation is presented in Section 2. After a section on notation, and the methodological Section 4, the principal graph-theoretic Proposition 5.1 appears in Section 5. The cycle structure of plane graphs is explored in Section 6, which ends with the proof of Theorem 1.3 (using Proposition 5.1). Sections 7 and 8 are devoted to the proof of Proposition 5.1.

The proof of Proposition 5.1 is a somewhat complicated graph-theoretic analysis of a number of possible cases. It is tempting to hope for a neater and more appetising proof of Theorem 1.3.

## 2. Application to site percolation

The percolation process is a prominent model for connectivity in a random medium. The model has emerged as central to the mathematical and physical theories of phase transition, and its theory is ramified and complex. Percolation comes in two flavours, bond and site, and it is site percolation that is relevant here. See [8] for an account of the standard theory of percolation.

Let $G = (V, E)$ be an infinite connected graph, and let $p \in [0, 1]$. Each vertex (or 'site') $v \in V$ is coloured *black* with probability $p$ and *white* otherwise, different vertices receiving independent colours. We write $\mathbb{P}_p$ for the corresponding probability measure. We choose some vertex, called the *origin*, and write $I$ for the event that the origin is the endpoint of some infinite black path. With $\theta(p) = \mathbb{P}_p(I)$, there exists a *critical probability* $p_c = p_c(G) \in [0, 1]$ such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c(G), \\ > 0 & \text{if } p > p_c(G). \end{cases} \tag{2.1}$$

The value of $p_c(G)$ is independent of the choice of origin.

The study of weak and strict inequalities for critical probabilities has a long history (see, for example, [14] and [16, Sect. 10]). A general method for proving strict inequalities for critical probabilities, and more generally for critical points of interacting systems, was described in [1]. One assumption for a naive application of this method is the quasi-transitivity of the underlying graph $G = (V, E)$. Recall that $G$ is *quasi-transitive* if its automorphism group acts on $V$ with only finitely many orbits.

Since $G$ is a subgraph of $G_*$, it is elementary that $p_c(G_*) \le p_c(G)$. Strict inequality is harder to prove. The following was proved in [10].

**Theorem 2.1** ([10, Thm 1.2])**.** *Let $G \in \mathcal{G}$ be quasi-transitive. Then $p_c(G_*) < p_c(G)$ if and only if $G$ has property $\Pi$.*

Using Theorems 2.1 and 1.3, one obtains the following application to percolation of the results of this article.

**Theorem 2.2.** *Let $G \in \mathcal{G}$ be quasi-transitive. The strict inequality $p_c(G_*) < p_c(G)$ holds if only if $G_\Delta$ is not a triangulation.*

This extends the earlier result of [10, Thm 1.4] which was restricted to transitive graphs, for which the proof is different and less complicated.

*Proof of Theorem 2.2 using Theorem 1.3.* Since $G$ is assumed quasi-transitive, we have that $G_\Delta$ is quasi-transitive and belongs to $\mathcal{G}$ (this is easily seen, and a formal statement with proof appears at Theorem 4.2(e)). Every infinite path of $G$ contains an infinite path of $G_\Delta$, and conversely an infinite path of $G_\Delta$ is an infinite path of $G$. Therefore, $G$ and $G_\Delta$ (respectively, their two matching graphs) have equal critical probabilities.

If $G_\Delta$ is a triangulation, then its matching graph is also $G_\Delta$, so that their critical probabilities are equal. Assume that $G_\Delta$ is not a triangulation. By Theorems 1.3 and 2.1, it suffices to show that $G_*$ has a $2\infty$-nst path. This is included in [11, Lemma 4.3(a)], and is given explicitly in Theorem 4.2(d). □

Non-self-touching paths were introduced in [1] where they were called 'stiff paths' (see also [3, 10] and [8, p. 66]).

Suppose $H$ is a connected, quasi-transitive graph. Let $N$ be the number of infinite black clusters of site percolation on $H$. It was proved in [12, 22] that there exists $p_{\mathrm{u}}(H) \in [0,1]$ such that

$$\mathbb{P}_p(N=1) = \begin{cases} 0 & \text{if } p < p_{\mathrm{u}}(H), \\ 1 & \text{if } p > p_{\mathrm{u}}(H). \end{cases}$$

Evidently, $p_{\mathrm{c}}(H) \le p_{\mathrm{u}}(H)$. Let $G \in \mathcal{G}$ be quasi-transitive. It is known that $p_{\mathrm{u}}(G) + p_{\mathrm{c}}(G_*) = 1$ (see [11, Thm 1.1]), and it follows by Theorem 2.2 that $p_{\mathrm{u}}(G)+p_{\mathrm{c}}(G) \ge 1$ with equality if and only if $G_\Delta$ is a triangulation.

## 3. Notation

A graph is denoted $G = (V, E)$ where $V$ is the vertex-set and $E$ the edge-set. Graphs considered here are mostly assumed to be countable (that is, finite or countably infinite), and simple (in that they have neither loops nor parallel edges); a possible exception to the last arises in the case of matching graphs, which may contain pairs of parallel diagonals created in abutting faces. An edge between vertices $u$, $v$ is denoted $\langle u, v\rangle$; if this edge exists, we say that $u$ and $v$ are *adjacent* and write $u \sim v$. The edge $\langle u, v\rangle$ is said to be *incident* to its endvertices. The *degree* of a vertex is the number of its incident edges, and $G$ is *locally finite* if all degrees are finite. Given $A, B \subseteq V$, $A$ is said to be *adjacent* to $B$, written $A \sim B$, if there exist $a \in A$ and $b \in B$ such that $a \sim b$.

A *walk* in $G$ is an alternating sequence $w = (\dots, w_0, e_0, w_1, e_1, \dots)$ where $w_i \in V$ and $e_i = \langle w_i, w_{i+1}\rangle \in E$ for all $i$; if $G$ is simple, the edges $e_i$ may be omitted from the definition. The walk $w$ is a *path* if the $w_i$ are distinct. The path $w$ is *non-self-touching* if $w_i \sim w_j$ if and only if $|i-j|=1$. A path $w$ is called a $2\infty$-*nst path* if it is doubly infinite and non-self-touching; we denote by $\mathrm{NST}(G)$ the set of all $2\infty$-nst paths of $G$. The graph-distance $d_G(u,v)$ between vertices $u$, $v$ is the minimal number of edges in paths from $u$ to $v$; for $A, B \subseteq V$; we set $d_G(A,B) = \min\{d_G(a,b) : a \in A,\ b \in B\}$. Two walks $\pi = (\pi_i)$, $\nu = (\nu_j)$ are said to be *non-touching* if $d_G(\pi_i, \nu_j) \ge 2$ for every pair $i$, $j$. A path from $u$ to $v$ is called a *geodesic* if it has exactly $d_G(u,v)$ edges. We note that a finite path is non-self-touching if it is a geodesic; a similar statement holds for infinite paths.

A *cycle* of $G$ is a finite walk of the form $w = (w_0, e_0, w_1, \dots, w_n)$ such that $w_0 = w_n$ and the sub-walk $(w_0, e_0, w_1, \dots, w_{n-1})$ is a path. Such a cycle has *length* $n$ and is called an $n$-*cycle*. The set of cycles of $G$ is denoted $\mathcal{C}(G)$.

Let $k \ge 1$. An infinite graph $G$ is called $k$-*connected* if, for all $v \in V$, there exist at least $k$ infinite paths starting from $v$ that are pairwise vertex-disjoint (except for their common starting point $v$). By Menger's theorem (see, for example, [2, Thm 1.1]), $G$ is $k$-connected if and only if, for all $v \in V$, there exists no set $A \subseteq V \setminus \{v\}$ of cardinality strictly less that $k$ whose removal leaves $v$ in a finite subgraph of $G$. For further discussion and references, see [2, Sect. 1] and [7, 13].

A graph $G = (V,E)$ is *planar* if it may be drawn in the plane in such a way that edges cross only at vertices. An embedded planar graph is called *plane*. A point $x \in \mathbb{R}^2$ is called a *vertex accumulation point* of $G$ if it is an accumulation point of $V$, and an *edge-accumulation point* if every neighbourhood of $x$ intersects some edge not incident with $x$. We shall consider only plane graphs with neither vertex- nor edge-accumulation points. The number of *ends* of a graph is the supremum of the number of infinite components obtained by deletion of finite sets of vertices.

A *face* of a one-ended, plane graph $G = (V,E)$ is a connected component of $\mathbb{R}^2 \setminus G$. By [18, Thm 3], if $G$ is 2-connected, the boundary of every face $F$ is a cycle of $G$, denoted $\partial F$. The *size* of the face $F$ is the number of edges in $\partial F$, and its (Euclidean) *diameter* is defined as

$$\mathrm{diam}(F) = \sup\{|x-y| : x, y \in F\}$$

where $|\cdot|$ denotes Euclidean distance. Let $C$ be a cycle of $G$, and write $\mathrm{int}(C)$ for the (open) bounded component of $\mathbb{R}^2 \setminus C$, and $\overline{C} = C \cup \mathrm{int}(C)$. We write $\mathrm{int}(C)$ also for the subgraph of $G$ obtained by deleting all vertices not belonging to $\mathrm{int}(C)$. A cycle $C$ is called *facial* if it is the boundary of some face.

We denote by $\mathcal{G}$ the set of countably infinite, 2-connected, locally finite, simple, plane graphs, embedded in the plane without vertex/edge-accumulation points, such that all faces have finite diameter (whence, in particular, such $G$ are one-ended).

We call a 3-cycle of $G$ a *separating triangle* if $\mathrm{int}(C)$ intersects one or more edges and/or vertices of $G$. For $G \in \mathcal{G}$, we write $G_\Delta$ for the subgraph of $G$ obtained by deleting any vertex/edge lying in the interior of any separating triangle of $G$. Thus

$G_\triangle$ has no separating triangle, and we say that $G_\triangle$ is $\triangle$-*empty*. We shall speak of $G_\triangle$ as being obtained from $G$ by 'emptying the separating triangles'. Since a $2\infty$-nst path of $G$ intersects the interior of no separating triangle, we have that

$$\mathrm{NST}(G) = \mathrm{NST}(G_\triangle). \tag{3.1}$$

The one-ended, plane graph $G$ is a *triangulation* if every face is bounded by a 3-cycle. Let $u, v \in V$ be such that $u \nsim v$ but there exists some face $F$ with $u, v \in \partial F$; we may choose to add to $F$ the further edge $\langle u, v\rangle$, and we call this a *diagonal* of $G$ (or of $G_*$), denoted $\delta(u, v)$.

The *matching graph* $G_*$ of $G \in \mathcal{G}$ is obtained from $G$ by adding all diagonals to all non-triangular faces. See Figure 1.1, and note that $G_*$ is not generally planar. We shall work also with the so-called 'facial graph' of $G$; see Section 4.3. The matching graph was introduced by Sykes and Essam [23] in the context of percolation theory.

The reasons for the assumption of 2-connectivity are as follows. Let $G$ be 1-connected but not 2-connected. Then there exist cutpoints $c$ such that $G \setminus \{c\}$ has one or more finite components. Such components cannot be relevant to the occurrence or not of property $\Pi$ since no $2\infty$-nst path (of either $G$ or $G_*$) may access them. Linked to this is the fact that site percolation on $G$ possesses an infinite cluster if and only $G \setminus \{c\}$ contains such a cluster. Moreover, as remarked above, 2-connectivity is needed for the faces of $G$ to be bounded by cycles.

**Remark 3.1.** *We close this section with a note about the distinction between planar and plane graphs. A planar graph $H$ is said to have property $\mathcal{N}$ if it possesses a $2\infty$-nst path. Evidently $\mathcal{N}$ is a graph property of $H$ which is independent of the choice of plane embedding. The situation for matching graphs is potentially more complicated since the diagonals of a plane graph depend on its facial structure and hence on its embedding. If $H$ is 3-connected, its embedding is unique in the sense of the cellular-embedding theorem of* [21, p. 42]*; see also* [11, Thm 2.1]. *Therefore, $\mathcal{N}$ is a graph property in this case.*

*The picture is more complicated if $H$ has connectivity 2. Assume this, and in addition* that $H$ is quasi-transitive. *Let $G \in \mathcal{G}$ be a plane embedding of $H$. By the proof of Theorem 8.25 in* [19, Sect. 8.8]*, there exists a 3-connected plane graph $G'$ from which $G$ is obtained by adding certain 'dangling loops'. Since $G'$ is 3-connected, by the cellular-embedding theorem its embedding is unique as above, so that every embedding of $H$ gives rise to the same $G'$. Furthermore, one sees from the relationship between $G$ and $G'$ that $G$ has $\mathcal{N}$ if and only if $G'$ has $\mathcal{N}$. In conclusion, for 2-connected,* quasi-transitive *planar graphs, property $\mathcal{N}$ is a graph property and is independent of the choice of plane embedding. We shall see in Theorem 4.2*(d) *that one such embedding, and hence all such embeddings, have property $\mathcal{N}$.*

## 4. Three techniques

4.1. **Oxbow removal.** Paths can fail to be non-self-touching through the existence of pairs of vertices that are not neighbours in the path but are neighbours in the graph. It is useful to have a method for extracting a non-self-touching path from a path containing such vertex-pairs. The method in question was used in [10], and is termed *oxbow removal*. We shall make use of the following extract from [10, Lemma 4.1(b)].

**Lemma 4.1.** *Let $H$ be a simple, plane graph embedded in $\mathbb{R}^2$. Let $\pi$ be a finite (respectively, infinite) path with endpoint $v$. There exists a non-empty subset $\pi'$ of the vertex-set of $\pi$ that forms a finite (respectively, infinite) non-self-touching path of $H$ starting at $v$. If $\pi$ is finite, then $\pi'$ may be chosen with the same endvertices as $\pi$.*

The related process of 'loop-erasure' is familiar in graph theory and probability; see, for example, [9, Sect. 2.2]. As noted in Section 3, a geodesic is non-self-touching. By Lemma 4.1, every locally finite, infinite, connected, simple graph possesses a singly infinite non-self-touching path.

*Proof.* Let $\pi = (v_0, v_1, v_2, \dots)$ be a path from $v = v_0$, either finite or infinite. We start at $v_0$ and move along $\pi$ in increasing order of vertex-index. Let $J$ be the least $j$ such that there exists $i \in \{0, 1, \dots, j-2\}$ with $v_i \sim v_J$, and let $I$ be the earliest such $i$. We delete from $\pi$ the subpath $(\pi_{I+1}, \dots, \pi_{J-1})$ (which is termed an *oxbow*), thus obtaining a new path $\pi_1$ starting at $v$. If $\pi$ is finite then $\pi_1$ has the same endvertices as $\pi$. This process is iterated until no oxbows remain. □

4.2. **Existence of 2∞-nst paths.** We present an elementary theorem concerning the existence of $2\infty$-nst paths. Recall the graph $G_\triangle$, obtained from $G$ by emptying all 3-cycles; see before Theorem 1.3.

Here is some notation. A face $F$ of $G \in \mathcal{G}$ satisfying $0 \notin \overline{F}$ is called $\zeta$-*acute* if there exists a sector $S$ of $\mathbb{R}^2$ with vertex 0 and angle $\zeta$ such that $F \subseteq S$.

**Theorem 4.2.**

(a) *Let $G$ be an infinite, connected, plane graph such that $G_\triangle$ is 4-connected. Then $G$ contains a $2\infty$-nst path.*
(b) *Every infinite, $\triangle$-empty, triangulation $T$ contains a $2\infty$-nst path.*
(c) *Let $G \in \mathcal{G}$. Suppose there exists $\zeta \in (0, \frac{1}{2}\pi)$ such that $F$ is $\zeta$-acute for all but finitely many faces $F$. Then $G$ and $G_*$ have $2\infty$-nst paths.*
(d) *If $G \in \mathcal{G}$ is quasi-transitive, then $G$ and $G_*$ have $2\infty$-nst paths.*
(e) *If $G \in \mathcal{G}$ is quasi-transitive, then $G_\triangle \in \mathcal{G}$ and $G_\triangle$ is quasi-transitive.*

The conditions of (a) and (c) are sufficient but evidently not necessary for the existence of a $2\infty$-nst path. Instances of non-self-touching paths are provided by

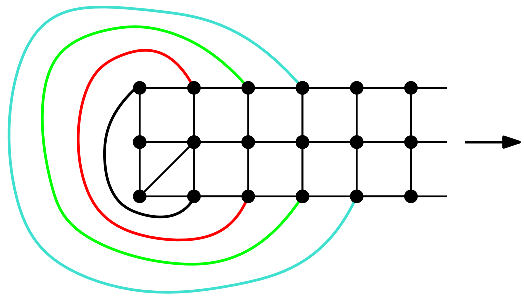

FIGURE 4.1. A 3-connected graph $G \in \mathcal{G}$ without separating triangles such that neither $G$ nor $G_*$ has a $2\infty$-nst path. Each vertex in the upper horizontal line is joined to the vertex one unit to its right in the lower line. A diagonal has been added to ensure the graph is truly 3-connected.

geodesics, and the existence of infinite geodesics has been explored in several articles including [5, 20, 24]. Figure 4.1 contains an illustration of a 3-connected $G \in \mathcal{G}$ such that neither $G$ nor its matching graph has a $2\infty$-nst path.

*Proof.* (a) Let $G = (V, E)$ be as stated. By the 4-connectedness of $G_\Delta$, for $v \in V$, there exist four infinite paths of $G_\Delta$ from $v$ that are pairwise vertex-disjoint except for the point $v$. Label these $\pi_i$ in a clockwise manner, and write $\pi_i^- = \pi_i \setminus \{v\}$. Then $d_{G_\Delta}(\pi_1^-, \pi_3^-) \geq 2$. For $i = 1, 3$, the path $\pi_i^-$ may be reduced by oxbow removal (see Lemma 4.1) to a singly infinite non-self-touching path $\nu_i$ with the same endvertex as $\pi_i$. The path $\nu := \nu_1 \cup \{v\} \cup \nu_3$ contains the required $2\infty$-nst path. On adding the contents of the original triangles back into $G_\Delta$, we see that $\nu$ is a $2\infty$-nst path of $G$.

(b) Let $T = (V, E)$ be as in the statement of the theorem. Since $T$ is $\triangle$-empty, it is 4-connected (see, for example, [4, Sect. 1] and [17, p. 91]), and the claim follows by part (a).

For the sake of completeness, we include a sketch proof of the 4-connectedness of such $T$. Suppose that $T$ is not 4-connected. It is standard that, as a triangulation, $T$ is 3-connected. Therefore, there exists $v \in V$ such that the maximum number of infinite paths from $v$ that are pairwise vertex-disjoint (except at $v$) is exactly 3. By Menger's theorem, there exists a triple $A = \{a, b, c\}$ of distinct vertices (with $v \neq a, b, c$) such that every infinite path from $v$ intersects $A$, and $A$ is minimal with this property.

Let $C$ be the (finite) connected component containing $v$ in the graph $T$ with $A$ deleted. Since $A$ is a minimal cutset, there exist $a', b', c' \in C$ such that $\langle a, a' \rangle, \langle b, b' \rangle, \langle c, c' \rangle$ are edges of $T$. Since $T$ is a maximal triangulation (in that no edge may be added to to $T$ without contradicting planarity), the edges $\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle$ exist in $T$. That is, $A$ is a separating triangle. By assumption, $T$ has no separating triangle, and therefore $T$ is 4-connected.

(c) We outline the proof, which is an adaptation of that of [10, Lemma 4.3(a)]. Suppose the condition holds, and let $L_\theta$ denote the singly infinite straight line from 0 inclined at clockwise angle $\theta$ to the $x$-axis $X$. Let $S_+$ be the closed sector between $L_0$ and $L_\zeta$ (clockwise), and let $I_+$ be the property that $G$ has some singly infinite path $\pi_+$ lying within $S_+$. If $I_+$ fails to hold, there exists a family $\mathcal{K}$ of arcs of $S_+$ ($\subseteq \mathbb{R}^2$), each with endpoints in $L_0$ and $L_\zeta$, such that (i) each $\kappa \in \mathcal{K}$ intersects no edge of $G$, and (ii) the Euclidean distances $d(0,\kappa)$ are unbounded as $\kappa$ ranges over $\mathcal{K}$. Each $\kappa \in \mathcal{K}$ lies in the interior of some face of $F$. Since there exist only finitely many faces that intersect both $L_0$ and $L_\zeta$, the statement $I_+$ must hold. Write $\nu_+$ for a non-self-touching path obtained from $\pi_+$ by oxbow removal (see Lemma 4.1).

By a similar argument with $S_+$ replaced by $S_- := -S$ (the sector bounded by $L_\pi$ and $L_{\pi+\zeta}$), $G$ has some singly infinite, non-self-touching path $\nu_-$ lying in $S_-$. Since $\pi - \zeta > \frac{1}{2}\pi > \zeta$, the set $\mathcal{A}$ of faces that intersect both $S_-$ and $S_+$ is finite. Find a shortest path $\pi$ of $G$ that connects $\nu_+$ and $\nu_-$ and intersects no $F \in \mathcal{A}$. The union $\nu_- \cup \pi \cup \nu_+$ contains (after oxbow removal) a $2\infty$-nst path.

The same argument applies to the matching graph $G_*$.

(d) Let $H$ be quasi-transitive, and consider its plane embeddings that belong to $\mathcal{G}$. By Remark 3.1, either all or no plane embeddings (respectively, their matching graphs) have $2\infty$-nst paths. Since $H$ is quasi-transitive, it may be embedded in either the Euclidean or hyperbolic plane (denoted $\mathcal{H}$) in such a way that its edges are geodesics and its automorphisms extend to isometries of the plane (see [19, Thm 8.25 and Sect. 8.8] and [11, Thm 2.1]); let $G \in \mathcal{G}$ be such an embedding of $H$ and consider for definiteness the hyperbolic case (in the model of the Poincaré disk — see [6] for an account of hyperbolic geometry). We note that, by the isometricity property, the diameters of faces of $G$ are bounded uniformly above (in the hyperbolic metric). The current claim is the content of [10, Lemma 4.3(a)].

(e) There is a partial order $\le$ on the set $\mathrm{ST}(G)$ of separating triangles of $G = (V,E)$ given by $T_1 \le T_2$ if $T_1 \subseteq \overline{T_2}$. A triangle $T \in \mathrm{ST}(G)$ is *maximal* if it is maximal with respect to $\le$, and $\mathcal{M}$ denotes the set of maximal triangles. Since $G$ is quasi-transitive without accumulation points, for $T' \in \mathrm{ST}(G)$, there exists $T \in \mathcal{M}$ with $T' \le T$. Since each $T \in \mathcal{M}$ is a subgraph of $G_\Delta$, and $G$ and $G_\Delta$ agree off the union of the maximal triangles, $G_\Delta$ has only bounded faces.

We show next that $G_\Delta$ is 2-connected. Let $v$ be a vertex of $G_\Delta$. Since $v \in V$ and $G$ is 2-connected, there exist infinite paths $\pi_1$, $\pi_2$ of $G$ that are vertex-disjoint except at their common initial vertex $v$. Let $T \in \mathcal{M}$. If $\pi_i$ intersects $T$, we find the first (respectively, last) intersection point $x$ (respectively, $y$), and we remove from $\pi_i$ the section of the path lying strictly between $x$ and $y$. This results in a subpath $\pi_i(T)$

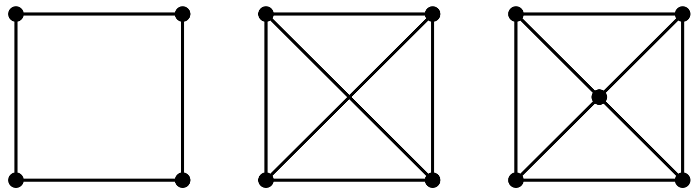

FIGURE 4.2. A square of the square lattice, its matching graph, and with its facial site added.

that does not intersect $\mathrm{int}(T)$. The process is iterated as $T$ ranges over $\mathcal{M}$, and the outcome is an infinite subpath $\nu_i$ of $\pi_i$ lying in $G_\Delta$. Therefore, $G_\Delta$ is 2-connected.

The quasi-transitivity of $G_\Delta$ follows from that of $G$, and the claim is proved. □

4.3. **The facial graph.** Let $G \in \mathcal{G} = (V, E)$, and let $\mathcal{Q}$ be the set of all non-triangular faces of $G$. We shall work with the graph $\widehat{G} = (\widehat{V}, \widehat{E})$ obtained from $G$ by adding a new vertex within each face $F \in \mathcal{Q}$, and adding an edge from every vertex in the boundary $\partial F$ to this central vertex. These new vertices are called *facial sites*, and the graph $\widehat{G}$ is called the *facial graph* of $G$. The facial site in the face $F$ is denoted $\phi(F)$. See [10], [16, Sect. 2.3], [19, Sect. 8.8], and also Figure 4.2. If $\langle v, w\rangle$ is a diagonal of the matching graph $G_*$, it lies in some face $F$ of $G$ with four or more edges, and we write $\phi(v, w) = \phi_F(v, w) = \phi(F)$ for the corresponding facial site.

Of importance in this work is the graph $\widehat{G}_\Delta = (\widehat{V}_\Delta, \widehat{E}_\Delta)$, defined as the graph obtained by emptying the separating triangles of the facial graph $\widehat{G}$. We note that $\widehat{G}_\Delta = (\widehat{G})_\Delta$ but generally $\widehat{G}_\Delta \neq \widehat{(G_\Delta)}$. The reason for this distinction lies in part (c) of the following lemma. We recall the set $\mathrm{NST}(H)$ of $2\infty$-nst paths of a graph $H$. By (3.1) applied to $\widehat{G}$, we have that

$$\mathrm{NST}(\widehat{G}) = \mathrm{NST}(\widehat{G}_\Delta). \tag{4.1}$$

**Lemma 4.3.** *Let $G = (V, E) \in \mathcal{G}$.*

(a) *Let $\nu \in \mathrm{NST}(G_*)$, and let $F$ be a face of $G$. If $\nu \cap \partial F \neq \varnothing$, then the intersection is exactly one of the following: (i) a single vertex of $G$, (ii) a single edge of $G$, (iii) a single diagonal of $G_*$. Moreover, the graph $\nu$ is plane.*

(b) *For $\nu \in \mathrm{NST}(G_*)$, let the path $\widehat{\nu} = \sigma(\nu)$ on $\widehat{G}$ be obtained from $\nu$ by replacing any diagonal $\delta(v, w)$ in a face $F$ by the pair $\langle v, \phi(F)\rangle, \langle \phi(F), w\rangle$ of edges. The function $\sigma$ maps $\mathrm{NST}(G_*)$ into $\mathrm{NST}(\widehat{G})$ and is an injection. The set $\mathrm{NST}(\widehat{G})$ may be expressed as the disjoint union*

$$\mathrm{NST}(\widehat{G}) = \sigma(\mathrm{NST}(G_*)) \cup \mathrm{NST}_2(\widehat{G}), \tag{4.2}$$

*where* $\mathrm{NST}_2(\widehat{G})$ *is the subset of* $\mathrm{NST}(\widehat{G})$ *containing all* $\widehat{\nu}$ *for which, for some face* $F$ *of* $G$*, we have (i)* $\phi(F) \notin \widehat{\nu}$*, and (ii) the intersection* $\widehat{\nu} \cap \partial F$ *contains a pair of non-adjacent vertices.*

(c) *Let* $\mathrm{ST}(H)$ *denote the set of separating triangles of a plane graph* $H$*. We have that* $\mathrm{ST}(G) \subseteq \mathrm{ST}(\widehat{G})$*, and moreover*

$$\mathrm{ST}(\widehat{G}) = \mathrm{ST}(G) \cup \mathrm{ST}_2(\widehat{G}), \tag{4.3}$$

*where* $\mathrm{ST}_2(\widehat{G})$ *is the set of all non-facial* 3*-cycles of* $\widehat{G}$ *comprising two edges of the form* $\langle u, \phi(F)\rangle$*,* $\langle v, \phi(F)\rangle$ *for some face* $F$ *of* $G$ *and some* $u, v \in \partial F$ *with* $d_{\partial F}(u, v) \geq 2$*, together with an edge* $\langle u, v\rangle$ *of* $G$*.*

(d) *Let* $\widehat{\nu}$ *be a finite non-self-touching path of* $\widehat{G}$*. There exists a subsequence of* $\widehat{\nu}$ *with the same endvertices that forms a non-self-touching path* $\nu$ *of* $G_*$*.*

We note some further notation. Firstly, the process used in the proof of (d), to replace $\widehat{\nu}$ by $\nu$, is termed *$\phi$-removal*. Secondly, since we shall be interested in the mapping $\sigma$, we introduce another binary relation on the vertex-set $\widehat{V}$ of $\widehat{G}$, namely:

(4.4) for $x, y \in \widehat{V}$, we write $x \mathrel{\widehat{\sim}} y$ if $G$ has some facial cycle $C$ such that $x, y \in \overline{C}$.

The negation of $\widehat{\sim}$ is written $\widehat{\nsim}$. For $x, y \in V$, we have $x \mathrel{\widehat{\sim}} y$ if and only if $x$, $y$ are neighbours in $G_*$.

*Proof.* (a) This was proved at [10, Lemma 4.4]. Such $\nu$ cannot contain three or more vertices of any given face since that would contradict the non-self-touching property. If $\nu$ contains two such vertices, it must contain the corresponding edge. If $\nu$ were non-planar, it would contain two or more diagonals of some face.

(b) That $\sigma$ is an injection into $\mathrm{NST}(\widehat{G})$ holds by (a) and the obvious invertibility of $\sigma$. Equation (4.2) holds by a consideration of $2\infty$-nst paths $\nu \in \mathrm{NST}(\widehat{G}_\Delta) \setminus \sigma(\mathrm{NST}(G_*))$.

(c) The inclusion holds since $G$ is a subgraph of $\widehat{G}$. Let $T \in \mathrm{ST}(\widehat{G}) \setminus \mathrm{ST}(G)$. Since $T \notin \mathrm{ST}(G)$, it contains some edge of the form $\langle u, \phi(F)\rangle$. Since it is a separating 3-cycle, it contains a further edge of the form $\langle v, \phi(F)\rangle$ where $d_{\partial F}(u, v) \geq 2$. The claim of (4.3) follows.

(d) Let $\widehat{\nu}$ be as given, and view it as a directed path. If $\widehat{\nu} \in \sigma(\mathrm{NST}(G_*))$, we simply replace the facial sites in $\widehat{\nu}$ by the corresponding diagonals. Assume that $\widehat{\nu} \in \mathrm{NST}_2(\widehat{G})$, and let $F$ be a face of $G$ such that $\phi(F) \notin \widehat{\nu}$ and $\widehat{\nu} \cap \partial F$ contains two (or more) non-adjacent vertices. Let $x$ (respectively, $y$) be the first (respectively, last) vertex of $\widehat{\nu}$ in $\partial F$, and note that $x \mathrel{\widehat{\sim}} y$. We delete from $\widehat{\nu}$ the subpath lying between $x$ and $y$ while retaining these two vertices and adding the corresponding edge (this edge lies in $E$ if $x \sim y$ in $G$, and is a diagonal otherwise). This process is iterated for each such face, and the ensuing path is as required. □

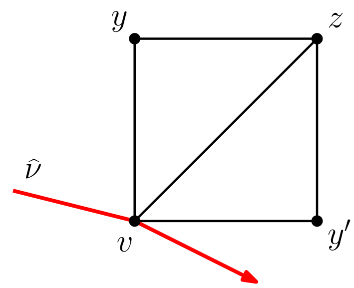


FIGURE 4.3. The 4-cycle in Proposition 5.1(b) comprises two triangles with a common edge

## 5. The main proposition

We present here the main proposition, which will be used twice in the proof of Theorem 1.3. The proof of the proposition is deferred to Sections 7 and 8.

**Proposition 5.1.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$.*

(a) *Let $F$ be a face of $G_\Delta$ with four or more edges, and let $\nu \in \mathrm{NST}(G_*)$ be a path that includes some vertex $v \in \partial F$. There exists $\overline{\nu} \in \mathrm{NST}(G_*)$ that includes some diagonal of $F$.*

(b) *Let $Q$ be a 4-cycle of $\widehat{G}_\Delta$ comprising the union of two triangles with a common edge $\langle v, z\rangle$ (as in Figure 4.3), and let $\widehat{\nu} \in \sigma(\mathrm{NST}(G_*))$ be a path that includes no facial site but includes $v$. Either there exists $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ that includes some facial site, or there exists $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ that includes no facial site but includes $z$.*

*Furthermore, the pair $\nu$, $\overline{\nu}$ (respectively, $\widehat{\nu}$, $\widehat{\nu}_1$) differ on only finitely many edges.*

## 6. Cycle structure of a plane graph

First, we explain how to define the so-called 'exterior cycle' of a cycle of a plane graph. This is followed by a description of a system of nested cycles surrounding a given cycle of a triangulation.

6.1. **Exterior cycles.** Let $G = (V, E) \in \mathcal{G}$, and recall the set $\mathcal{C} = \mathcal{C}(G)$ of cycles of $G$. For $A \in \mathcal{C}(G)$, we shall construct a new cycle $B = \mathrm{Ext}(A)$ called the *exterior cycle* of $A$. An intuitive explanation of this is as follows (see Figure 6.1). As we walk around the cycle $A$, we may encounter 'shortcuts' using no edge of $\mathrm{int}(A)$ — these are edges not in $A$ that join two vertices of $A$. When allowing the walker to take such shortcuts, the exterior cycle is the walker's route that traverses fewest edges (including shortcuts).

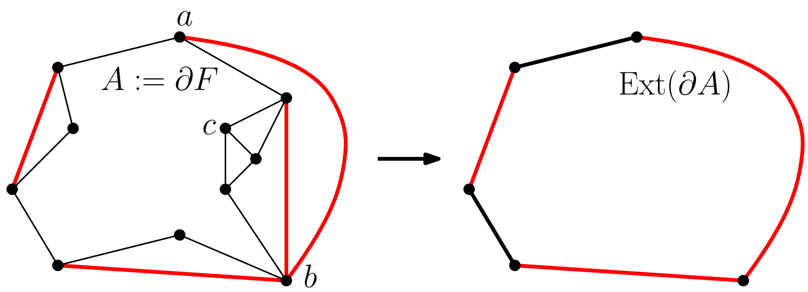


FIGURE 6.1. *Left*: A face $F$ of $G_\Delta$ surrounded by the (black) cycle $A := \partial F$, and with further edges in $Y$ coloured red. *Right*: The exterior cycle $\mathrm{Ext}(\partial F)$.

Let $A \in \mathcal{C}(G)$, let $X$ be the set of edges of $G$ of the form $f = \langle a, b\rangle$ with $a, b \in A$; in particular, the edges of $A$ lie in $X$. Let $Y$ be the subset of $X$ containing edges that neither lie in nor intersect $\mathrm{int}(A)$. Recalling that $G$ is embedded in the plane, an edge $f = \langle a, b\rangle \in Y$ may appear either clockwise or anticlockwise around $A$ (in that, when considered as a directed edge from $a$ to $b$, it has two distinct possible placements in the embedding). Consider the subgraph of $G$ with edge-set $X$ and its incident vertices, denoted as $X$ also. Then $X$ has an outer cycle formed of edges in $Y$, and we write $B = \mathrm{Ext}(A)$ for this cycle. Note that the number of edges in $B$ is no greater than the number in $A$. Furthermore, if $A$ is a 3-cycle, then $A = \mathrm{Ext}(A)$. The construction is illustrated in Figure 6.1 in the case when $A$ is facial in $G$.

**Remark 6.1.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$, and let $A$ be a cycle of $G_\Delta$ (and hence of $G$ also). We may use either $G$ or $G_\Delta$ in constructing $\mathrm{Ext}(A)$, and the outcome is the same. If, in addition, $A$ is a facial cycle of $G$ (that is, $A = \partial F$ for some face $F$ of $G$), then $B := \mathrm{Ext}(A)$ is a cycle of $\widehat{G}_\Delta$ whose interior contains only one facial site and its incident edges. We may denote this facial site $\phi(F)$.*

**Lemma 6.2.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$. Let $F$ be a face of $G_\Delta$ (and hence of $G$ also) with size 4 or more.*

- (a) *The exterior cycle $\mathrm{Ext}(\partial F)$ is a cycle of $G_\Delta$ with length 4 or more.*
- (b) *Let $G(F)$ (respectively, $G_\Delta(F)$) be obtained from $G$ (respectively, $G_\Delta$) by removing all vertices and incident edges within $\mathrm{int}(\mathrm{Ext}(\partial F))$. Then $\mathrm{NST}(G_\Delta) = \mathrm{NST}(G_\Delta(F))$.*
- (c) *Let $\widehat{G}(F)_\Delta$ be obtained from the facial graph $\widehat{G}(F)$ of $G(F)$ by emptying its separating triangles (that is, $\widehat{G}(F)_\Delta := (\widehat{G(F)})_\Delta$). Then $\widehat{G}(F)_\Delta = \widehat{G}_\Delta$. In particular, $\mathrm{NST}(\widehat{G}_\Delta) = \mathrm{NST}(\widehat{G}(F)_\Delta)$.*

**Remark 6.3.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$. By Lemma 6.2(a), the exterior cycle of a 4-cycle $Q$ of $G_\Delta$ is $Q$ itself.*

*Proof of Lemma 6.2.* (a) The length $l$ of the cycle $\mathrm{Ext}(\partial F)$ satisfies $l \geq 1$. Evidently, $l \geq 3$ since $G$ is simple. If $l = 3$, then $\mathrm{Ext}(\partial F)$ is a 3-cycle of $G_\Delta$ whose interior intersects $\partial F$, in contradiction of the definition of $G_\Delta$.

(b) Let $\pi \in \mathrm{NST}(G_\Delta) \setminus \mathrm{NST}(G_\Delta(F))$, so that $\pi$ contains some vertex, $c$ say, in $\mathrm{int}(\mathrm{Ext}(\partial F))$. Thinking of $\pi$ as a directed path, let $a$ be the last vertex of $\pi \cap \mathrm{Ext}(\partial F)$ prior to $c$, and $b$ the first vertex of $\pi \cap \mathrm{Ext}(\partial F)$ after $c$ (see Figure 6.1). By the definition of exterior cycle, and the fact that $\partial F$ is a facial cycle, it is the case that $a$ and $b$ are neighbours in $\mathrm{Ext}(\partial F)$. Therefore $\pi \notin \mathrm{NST}(G_\Delta)$, a contradiction, whence $\mathrm{NST}(G_\Delta) \subseteq \mathrm{NST}(G_\Delta(F))$. Conversely, any $\pi \in \mathrm{NST}(G_\Delta(F)) \setminus \mathrm{NST}(G_\Delta)$ must contain non-consecutive vertices $a, b$ lying in $\mathrm{Ext}(F)$ such that $\mathrm{int}(G_\Delta(F))$ contains a path joining $a$ and $b$. As in the above, this requires that $a$ and $b$ are adjacent in $\mathrm{Ext}(\partial F)$, a contradiction.

(c) The graphs $G$ and $G(F)$ differ only on the interior of $\mathrm{Ext}(F)$. Therefore, the same holds for their facial graphs $\widehat{G}$ and $\widehat{G}(F)$. After emptying separating triangles, each of the two interiors of $F$ in the two resulting graphs is a wheel with a hub at the facial site $\phi(F)$ (recall Remark 6.1) and spokes to the vertices of $\mathrm{Ext}(F)$. It follows that $\widehat{G}(F)_\Delta = \widehat{G}_\Delta$ as claimed. □

6.2. **Cycle structure of a triangulation.** Let $H \in \mathcal{G}$ be a triangulation. For $A \in \mathcal{C}(H)$, we write $N_A$ for the set of neighbours of members of $A$ lying in the unbounded component of $\mathbb{R}^2 \setminus A$. Thus, $A \cap N_A = \varnothing$ and (since $G$ is a triangulation) every $a \in A$ has some neighbour $b \in N_A$. We think of the edges between $A$ and $N_A$ as ordered cyclically as one traverses $A$ clockwise.

The following lemma and more was proved in [15, Sect. 3], from which we extract the element of current interest.

**Lemma 6.4.** *Let $H \in \mathcal{G}$ be a triangulation, and let $A \in \mathcal{C}(H)$. The set $N_A$ contains a cycle $B \in \mathcal{C}(H)$ satisfying $A \subseteq \mathrm{int}(B)$ and $N_A \subseteq \overline{B}$.*

*Proof.* Consider the finite graph $J$ induced by the vertices of $H$ in $\overline{A} \cup N_A$. By construction, $J$ is connected, and is an inner triangulation (in that it is finite and all its faces except possibly its exterior face are triangles). The exterior face is the unique unbounded face, and it has a boundary $B$ comprising edges of $J \setminus \overline{A}$. The set $B$ forms a cycle since, if not, $J$ contains some $c$ such that $c \notin \overline{A}$ and $d_H(A, c) \geq 2$. This would be a contradiction. □

There follow two lemmas that will be used in the proof of Theorem 1.3 at the end of this section. Recall from Lemma 4.3 the map $\sigma : \mathrm{NST}(G_*) \to \mathrm{NST}(\widehat{G})$.

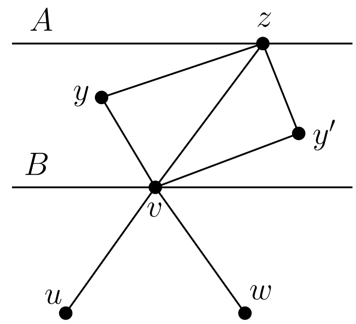


FIGURE 6.2. When $v \in \widehat{\nu} \cap B$, there exists $z \in A$ such that $v \sim z$ in $\widehat{G}_\Delta$. The edge $\langle v, z\rangle$ lies in two triangles whose union forms the quadrilateral illustrated here. Each of the vertices $y$, $y'$ may lie in either $A$ or $B$ or neither.

**Lemma 6.5.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$, and let $A$ be a cycle of the triangulation $\widehat{G}_\Delta$. Assume $G_*$ has a $2\infty$-nst path $\nu$ such that $\widehat{\nu} = \sigma(\nu)$ has the following properties: (i) $\widehat{\nu}$ includes no facial site, (ii) $\widehat{\nu} \cap N_A \ne \varnothing$, and (iii) $\widehat{\nu} \cap A = \varnothing$. There exists $\overline{\nu} \in \mathrm{NST}(G_*)$ such that either (i) $\overline{\nu}$ traverses some diagonal, or (ii) $\overline{\nu}$ traverses no diagonal but satisfies $\widehat{\nu} \cap A \ne \varnothing$. Furthermore, $\nu$ and $\overline{\nu}$ differ on only finitely many edges.*

*Proof of Lemma 6.5 using Proposition 5.1(b).* Let $\nu \in \mathrm{NST}(G_*)$ be as given. Since $\widehat{\nu} \cap N_A \ne \varnothing$ by assumption, we have that $\widehat{\nu} \cap B \ne \varnothing$ also (where $B$ is given in Lemma 6.4 with $H = \widehat{G}_\Delta$). Let $v$ $(\in V)$ be the first point in $\widehat{\nu}$ (considered as a directed path) that lies in $B$.

Since $B \subseteq N_A$, there exists an edge $e = \langle v, z\rangle$ of $\widehat{G}_\Delta$ with $z \in A$. The edge $e$ lies in two 3-cycles of $\widehat{G}_\Delta$, and the union of these triangles forms a quadrilateral $Q$ with $v$ and $z$ as opposite vertices. See Figure 6.2. The claim follows by Proposition 5.1(b). □

**Lemma 6.6.** *Let $G \in \mathcal{G}$ satisfy $G_\Delta \in \mathcal{G}$, and let $A$ be a cycle of $G_\Delta$ (and hence of $G$ also) of size 4 or more. If $G_*$ has some $2\infty$-nst path $\nu$, then either (i) there exists $\overline{\nu} \in \mathrm{NST}(G_*)$ that traverses some diagonal, or (ii) there exists $\overline{\nu} \in \mathrm{NST}(G_*)$ that traverses no diagonal but includes some vertex of $A$. Furthermore, $\nu$ and $\overline{\nu}$ differ on only finitely many edges.*

*Proof of Lemma 6.6 using Proposition 5.1(b).* Let $A' = \mathrm{Ext}(A)$ be the exterior cycle of $A$. By iteration of Lemma 6.4 applied to the triangulation $\widehat{G}_\Delta$, there exists a sequence $A_0, A_1, A_2, \dots$ of cycles in $\widehat{G}_\Delta$ such that $A_0 = A'$ and, for $i \ge 0$, $A_i \subseteq$

$\mathrm{int}(A_{i+1})$ and $A_{i+1} \subseteq N_{A_i} \subseteq \overline{A_{i+1}}$. Since $G \in \mathcal{G}$ and $A_i \subseteq \mathrm{int}(A_{i+1})$,

$$V \cap \mathrm{int}(A_i) \uparrow V \qquad \text{as } i \to \infty. \tag{6.1}$$

Let $\nu \in \mathrm{NST}(G_*)$ and $\widehat{\nu} = \sigma(\nu)$. If $\nu$ traverses some diagonal, we may take $\overline{\nu} = \nu$. Assume that $\nu$ traverses no diagonal, so that $\widehat{\nu}$ includes no facial site.

By (6.1), there exists $I$ such that $\widehat{\nu} \cap A_I \neq \varnothing$, and we pick $I = I(\widehat{\nu})$ minimal with this property. If $I = 0$, there is nothing to prove since $A_0 = A'$ and $A' \subseteq A$. Assume then that $I \geq 1$, and let $v \in \widehat{\nu} \cap A_I$; since $\widehat{\nu}$ includes no facial site, we have $v \in V$. By Lemma 6.5, there exists $\nu' \in \mathrm{NST}(G_*)$ such that either (i) $\nu'$ traverses some diagonal, or (ii) $\nu'$ traverses no diagonal but $\widehat{\nu}' := \sigma(\nu')$ satisfies $\widehat{\nu}' \cap A_{I-1} \neq \varnothing$. If (i) holds, the proof is complete. Otherwise, $\widehat{\nu}'$ satisfies $I(\widehat{\nu}') \leq I - 1$.

We continue by iteration. At each stage we could possibly obtain some $\overline{\nu} \in \mathrm{NST}(G_*)$ that traverses some diagonal. If this occurs at no stage of the iteration, we obtain finally some $\nu'' \in \mathrm{NST}(G_*)$ that traverses no diagonal, and such that $\widehat{\nu}'' = \sigma(\nu'')$ satisfies $I(\widehat{\nu}'') = 0$ and $\widehat{\nu}'' \cap A' \neq \varnothing$. The claim follows since $A' \subseteq A$. □

*Proof of Theorem 1.3 using Proposition 5.1.* Let $G \in \mathcal{G}$ be such that $G_\Delta \in \mathcal{G}$ is not a triangulation, and let $F$ be a face of $G_\Delta$ of size 4 or more. Let $\nu \in \mathrm{NST}(G_*)$. If $\nu$ traverses some diagonal then the proof is complete, so we may assume that $\nu$ traverses no diagonal. By Lemma 6.6, there exists $\nu_1 \in \mathrm{NST}(G_*)$ that traverses no diagonal but intersects $\partial F$. We apply Proposition 5.1(a) to complete the proof. □

## 7. Proof of Proposition 5.1(a)

We begin with an outline. Let $G$ be as in the statement. Since $G_\Delta$ is not a triangulation, it has some face $F$ of size 4 or more (note that $F$ is also a face of $G$). Let $\nu$ and $v$ be as in the statement of part (a). We shall explain how to make local changes to $\nu$ to obtain a $2\infty$-nst path $\overline{\nu}$ of $G_*$ that agrees with $\nu$ except on finitely many edges, and that contains some diagonal of $\partial F$. This will be done in the universe of non-self-touching paths on the facial graph $\widehat{G}_\Delta = (\widehat{V}_\Delta, \widehat{E}_\Delta)$. Let $\widehat{\nu} = \sigma(\nu)$ be the $2\infty$-nst path of $\widehat{G}_\Delta$ corresponding to $\nu$ (see Lemma 4.3(b) and (4.1)). We shall make local changes to $\widehat{\nu}$ to obtain a $2\infty$-nst path $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ that includes the facial site $\phi(F)$. The path $\overline{\nu} = \sigma^{-1}(\widehat{\nu}_1) \in \mathrm{NST}(G_*)$ has the required property. There are a number of steps in the pursuit of this strategy, as follows.

Let $G = (V, E)$, $v$, $F$ be as above, and let $\nu = (\dots, \nu_{-1}, \nu_0, \nu_1, \dots) \in \mathrm{NST}(G_*)$ with $\nu_0 = v$; it is sometimes convenient to think of $\nu$ as a *directed* path. We may assume that

$$\nu \text{ contains no diagonal of } F, \tag{7.1}$$

since otherwise there is nothing to prove. Therefore, by Lemma 4.3(a),

$$\nu \cap \partial F \text{ comprises either a single vertex of } V \text{ or a single edge of } E. \tag{7.2}$$

Rather than working with the boundary cycle $\partial F$ of the face $F$, we shall work with its exterior cycle $E = \mathrm{Ext}(\partial F)$. The latter cycle is facial (in both $G(F)$ and $G(F)_\Delta$, recall Lemma 6.2(b)) and we denote this face by $F' = \mathrm{int}(E)$, so that $E = \partial F'$. Recall from Lemma 6.2(c) that $\widehat{G}(F)_\Delta = \widehat{G}_\Delta$. By Remark 6.1, we may take $\phi(F') = \phi(F)$. Let $\widehat{\nu} = \sigma(\nu) \in \mathrm{NST}(\widehat{G}_\Delta)$. By (7.1) and (7.2),

(7.3) $$\widehat{\nu} \text{ does not contain the facial site } \phi(F'),$$

(7.4) $$\widehat{\nu} \cap \partial F' \text{ comprises either a single vertex of } V \text{ or a single edge of } E.$$

In the various steps and figures that follow, we write

$$u = \nu_{-1}, \quad v = \nu_0, \quad w = \nu_1.$$

Represent the triple $u$, $v$, $w$ in the plane graph $\widehat{G}_\Delta$ as in Figure 7.1, so that $F'$ lies 'above' the triple ($F'$ is depicted in the figure with its facial site and incident edges removed). Let $f_i = \langle v, y_i \rangle$, $i = 1, 2, \dots, r$, be the edges of $\widehat{G}_\Delta$ incident to $v$ in the sector obtained by rotating $\langle u, v \rangle$ clockwise about $v$ until it coincides with $\langle w, v \rangle$; the $f_i$ are listed in clockwise order. Since $\widehat{G}_\Delta$ is simple, the $y_i$ are distinct.

For a (directed) path $\pi$ and a vertex $x \in \pi$, let $\pi(x-)$ (respectively, $\pi(x+)$) be the subpath of $\pi$ prior to and including $x$ (respectively, after and including $x$).

There are two cases to consider depending on which case of (7.4) holds (see Sections 7.1 and 7.2). If $\widehat{\nu} \cap \partial F'$ is a singleton $v$ (as in Figure 7.1), we denote by $y_N$ and $y_{N+1}$ the two neighbours of $v$ lying in $\partial F'$ (in particular, we have $y_N \ne y_{N+1}$). If $\widehat{\nu} \cap \partial F'$ is an edge, we may take that edge to be $\langle v, w \rangle$, and we denote by $y_N$ the vertex of $\partial F'$ other than $w$ that is incident to $v$ (as in Figure 7.8); in this case we have $N = r$.

**Lemma 7.1.**

(a) *Let $s_0 = u$, $s_{r+1} = w$, and $s_i = y_i$ for $i = 1, 2, \dots, r$. If $s_i \sim s_j$ then $|i-j| = 1$. Conversely, $s_0 \sim s_1 \sim \cdots \sim s_N$ and $s_{N+1} \sim \cdots \sim s_{r+1}$, where $N$ is such that $y_N$ and $y_{N+1}$ are the two neighbours of $v$ lying in $\partial F'$.*

(b) *If $y_i \in \partial F'$, then $i \in \{N, N+1\}$.*

(c) *No $y_i$ lies in $\widehat{\nu}(u-) \cup \widehat{\nu}(w+)$.*

*Proof.* (a) Suppose $s_i \sim s_j$ where $j \ge i + 2$. Then $(v, s_i, s_j)$ forms a 3-cycle $T$ of the triangulation $\widehat{G}_\Delta$ whose interior intersects the edge $\langle v, s_{i+1} \rangle$. This is a contradiction since $\widehat{G}_\Delta$ is $\triangle$-empty. The partial converse holds as stated since $\widehat{G}_\Delta$ is a triangulation.

(b) This holds by the definition of the exterior cycle $F' = \mathrm{Ext}(F)$.

(c) This follows from the fact that $\nu$ is non-self-touching in $G_*$. □

For $i = 1, 2, \dots, r$, denote the neighbours of $y_i$ other than possibly $u, v, y_{i-1}, y_{i+1}, w$ as $z_{i,1}, z_{i,2}, \dots, z_{i,\delta_i}$, listed in clockwise order of the planar embedding. More precisely,

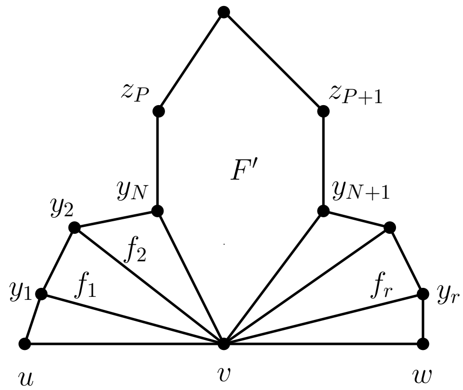


FIGURE 7.1. The path $\widehat{\nu}$ passes through a vertex $v$ that lies in the boundary of a 6-face $F'$. This is an illustration of the case when neither $\langle u, v\rangle$ nor $\langle v, w\rangle$ lie in $\partial F'$.

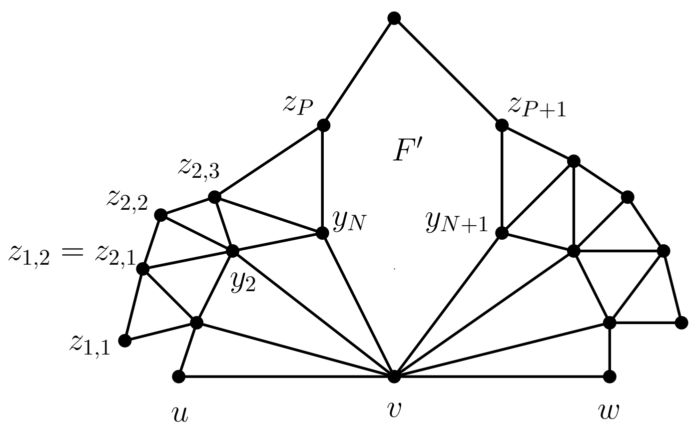


FIGURE 7.2. An illustration of the vertices $z_{i,j}$. Note that $z_{i,\delta_i} = z_{i+1,1}$ for $i \neq N$, as in (7.7), and that any two consecutive $z_{i,j}$ (other than $(z_P, z_{P+1})$) close a triangle, as in (7.6). This illustration is a simplification — see Figure 7.3.

we list the edges exiting $y_i$ (other than any edges to $u, v, y_{i-1}, y_{i+1}, w$) according to clockwise order, and we denote the other endvertex of the $j$th such edge as $z_{i,j}$. Note that, while the $z_{i,1}, z_{i,2}, \ldots, z_{i,\delta_i}$ are distinct for given $i$ (since $\widehat{G}_\Delta$ is simple), there may generally exist values of $i \neq j$ and $1 \leq a \leq \delta_i$, $1 \leq b \leq \delta_j$ with $z_{i,a} = z_{j,b}$.

We list the labels $z_{i,j}$ in lexicographic order (that is, $z_{a,b} < z_{c,d}$ if either $a < c$, or $a = c$ and $b < d$) as $z_1 < z_2 < \cdots < z_s$; this is a total order of the *label-set* but not necessarily of the underlying *vertex-set* since a given vertex may occur multiple

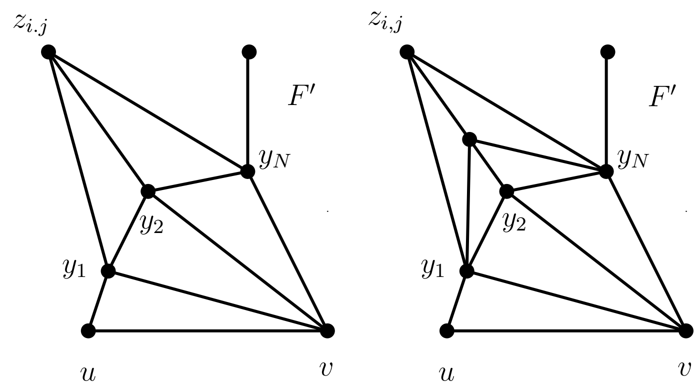


FIGURE 7.3. On the left, there is a vertex $z_{i,j}$ connected to each of $y_1, y_2, \dots, y_N$. On the right, the relationship of this vertex to $y_2$ is more complicated.

times (if two labelled vertices $z_a$, $z_b$ satisfy $z_a = z_b$ when viewed as vertices, we say that each label is an *image* of the other). If $a < b$ we speak of $z_a$ as preceding, or being to the *left* of $z_b$ (and $z_b$ succeeding, or being to the *right* of $z_a$). For $1 \le i \le r$, let

$$S_i = (z_{i,j} : j = 1, 2, \dots, \delta_i), \text{ viewed as an ordered subsequence of } Z. \tag{7.5}$$

Since $\widehat{G}_\Delta = (\widehat{V}_\Delta, \widehat{E}_\Delta)$ is a triangulation,

$$\langle z_{i,j}, z_{i,j+1} \rangle \in \widehat{E}_\Delta, \qquad j = 1, 2, \dots, \delta_i - 1, \ 1 \le i \le r, \tag{7.6}$$

and moreover

$$z_{i,\delta_i} = z_{i+1,1} \qquad 1 \le i < r, \ i \ne N, \tag{7.7}$$

whenever the relevant pair of vertices is defined.

As in Figure 7.2, let $y_N, y_{N+1}$ be the two neighbours of $v$ in $\partial F'$, and $z_P$, $z_{P+1}$ their further neighbours in $\partial F'$ (if $F'$ is a quadrilateral, we have $z_P = z_{P+1}$). It can be the case that $z_i \in \partial F'$ for some $i \notin \{P, P+1\}$.

See Figures 7.2 and 7.3 for illustrations of the $z_{i,j}$. By (7.6)–(7.7),

$$\pi_u = (u, z_1, z_2, \dots, z_P) \text{ and } \pi_w = (z_{P+1}, \dots, z_s, w) \text{ are walks of } \widehat{G}_\Delta. \tag{7.8}$$

Note that $\pi_u$ and $\pi_w$ may contain cycles and oxbows, and may intersect one another. Further information concerning the relationship between the $z_i$ and the $y_j$ may be gleaned from [15, Sect. 3].

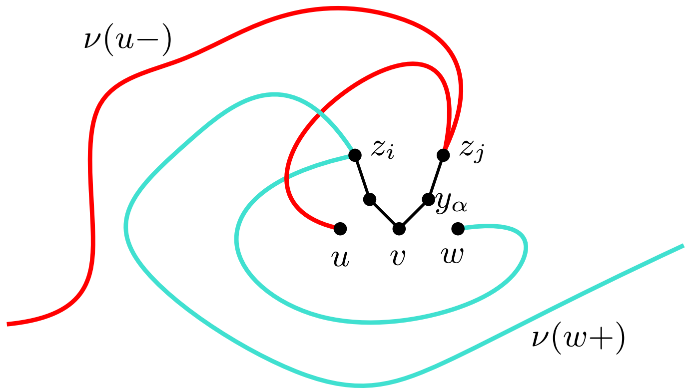


FIGURE 7.4. If $z_i \in \widehat{\nu}(w+)$ and $z_j \in \widehat{\nu}(u-)$ where $i < j$, then the pair $\widehat{\nu}(u-)$, $\widehat{\nu}(w+)$ fails to be non-touching, and is indeed intersecting.

In making changes to the path $\widehat{\nu}$, it is useful to first record which vertices lie in either $\widehat{\nu}(u-)$ or $\widehat{\nu}(w+)$, or in neither. We label each vertex $x \in V$ by

$$\begin{cases} U & \text{if } x \in \widehat{\nu}(u-), \\ W & \text{if } x \in \widehat{\nu}(w+), \\ Q & \text{if } x \notin \widehat{\nu}(u-) \cup \widehat{\nu}(w+). \end{cases}$$

Write $N_L$ be the number of $z_i$ with label $L$. Since $\nu \in \mathrm{NST}(G_*)$, by (7.1)

$$\text{every } x \in \partial F' \text{ satisfying } x \neq v, w \text{ is labelled } Q. \tag{7.9}$$

According to (7.2) there are two cases, which we consider in order.

7.1. **Case I: Suppose $\partial F'$ contains neither of the edges $\langle u, v\rangle$, $\langle v, w\rangle$.** This case is illustrated in Figure 7.1. Here is a technical lemma.

**Lemma 7.2.** *Suppose $N_U \geq 1$ and let $z_\rho = z_{\alpha,\beta}$ be the rightmost $z_i$ with label $U$. Let $\widehat{\nu}''_\rho(u-)$ be the subpath of $\widehat{\nu}(u-)$ from $z_\rho$ to $u$, and $\widehat{\nu}'_\rho(u-)$ that obtained from $\widehat{\nu}(u-)$ by deleting $\widehat{\nu}''_\rho(u-)$ while retaining its endpoint $z_\rho$.*

(a) *The path $\widehat{\nu}''_\rho(u-)$ moves around $v$ in an anticlockwise direction in the sense that the directed cycle $D$ obtained by traversing $\widehat{\nu}''_\rho(u-)$ from $z_\rho$ to $u$, followed by the edges $\langle u, v\rangle$, $\langle v, y_\alpha\rangle, \langle y_\alpha, z_\rho\rangle$, has winding number $-1$. Furthermore, $\overline{D} \cap \widehat{\nu}(w+) = \varnothing$.*

(b) *For $1 \leq i \leq \rho - 1$, $z_i$ is labelled either $Q$ or $U$.*

(c) *For $1 \leq i \leq \rho - 1$, $z_i$ has no $\widehat{G}_\Delta$-neighbour lying in $\widehat{\nu}'_\rho(u-)$, apart possibly from $z_\rho$ or one of its images. Moreover, for all $x \in \widehat{\nu}'_\rho(u-) \setminus \{z_\rho\}$, we have that $z_i \mathrel{\widehat{\not\sim}} x$.*

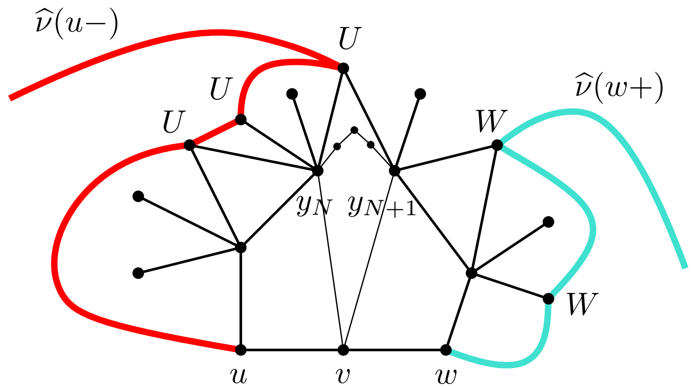


FIGURE 7.5. The path $\widehat{\nu}(u-)$ intersects the $z_a$ at the rightmost $z_\rho$ and then progresses anticlockwise to $u$. Similarly $\widehat{\nu}(w+)$ hits at the leftmost vertex $z_\lambda$ and progresses clockwise to $w$.

(d) *For* $1 \le i \le \rho$, $z_i$ *has no* $\widehat{G}_\Delta$*-neighbour lying in* $\widehat{\nu}(w+)$. *Moreover, for all* $x \in \widehat{\nu}(w+)$, *we have that* $z_i \mathrel{\widehat{\nsim}} x$.

*Proof.* (a) If the given cycle has winding number 1, then $\widehat{\nu}''_\rho(u-)$ intersects $\widehat{\nu}(w+)$ in contradiction of the definition of $\widehat{\nu}$. See Figure 7.4. The final claim holds since $v, y_N \notin \widehat{\nu}(w+)$.

(b) Let $1 \le i \le \rho-1$. If $z_i \in \widehat{\nu}(w+)$, then (as illustrated in Figure 7.4), $\widehat{\nu}(u-)$ and $\widehat{\nu}(w+)$ must intersect (when viewed as arcs in $\mathbb{R}^2$). This contradicts the planarity of $\widehat{\nu}$. Therefore, such $z_i$ is labelled either $Q$ or $U$.

(c) Let $1 \le i \le \rho - 1$ and suppose $z_i$ has a $\widehat{G}_\Delta$-neighbour $x$ (with $x$ a different vertex from $z_\rho$) belonging to $\widehat{\nu}'_\rho(u-)$. By a consideration of the cycle $D$ of Lemma 7.2(a), we have that $d_{\widehat{G}_\Delta}(x, \widehat{\nu}''_\rho(u-)) \le 1$, which (as above) contradicts the fact that $\widehat{\nu}(u-)$ is non-self-touching in $\widehat{G}_\Delta$. The second statement holds similarly, since $\nu \in \mathrm{NST}(G_*)$.

(d) This is similar to (c) above. □

Similar conclusions hold with $U$ replaced by $W$, and $z_\rho$ replaced by the leftmost $z_\lambda$ in $\widehat{\nu}(w+)$. See Figure 7.5 for an illustration of Lemma 7.2(b) and some of its consequences. In its approach towards $u$ (from infinity) $\widehat{\nu}(u-)$ passes through the rightmost $z_\rho$. It may subsequently visit one or more $z_i$ with $i < \rho$, but it must do this in decreasing order of suffix. Similarly, $\widehat{\nu}(w+)$ passes through the leftmost $z_\lambda$ and may subsequently visit one or more $z_i$ with $i > \lambda$ in clockwise order of suffix. That $\lambda > \rho$ (when these suffices are defined) holds by Lemma 7.2(b).

Let $z_\rho = z_{a,b}$ be the rightmost $z_i$ labelled $U$ (with $\rho = 0$ and $z_0 := u$ if $N_U = 0$). Similarly, let $z_\lambda = z_{c,d}$ be the leftmost $z_i$ labelled $W$ (with $\lambda = r + 1$ and $z_{r+1} := w$

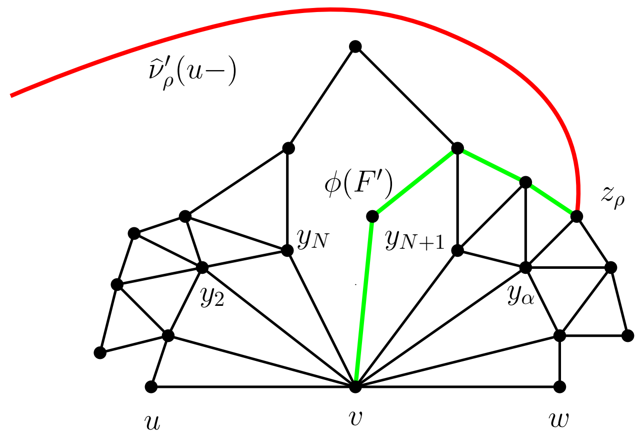


FIGURE 7.6. An illustration of $\widehat{\nu}_1$ in case (A), when the rightmost $z_i$ labelled $U$ is to the right of $z_{P+1}$.

if $N_W = 0$). By the non-self-touching property of $\nu$ (see also (7.9)), we have

$$z_\rho \mathrel{\widehat{\not\sim}} z_\lambda,\ \ z_\rho, z_\lambda \notin \partial F'. \tag{7.10}$$

For the special case when $\rho = 0$ and $\lambda = r+1$, we use here the fact that $v$ is not a facial site.

(A). *Suppose* $\rho \geq P+1$. Let $\alpha = \max\{i \geq N+1 : z_\rho \in S_i\}$, say $z_\rho = z_{\alpha,\beta}$ (recall the set $S_i$ from (7.5)). We add to $\widehat{\nu}'_\rho(u-)$ the set of vertices

$$W := \{z_{a,b} : N+1 \leq a \leq \alpha - 1,\ 1 \leq b \leq \delta_a\} \cup \{z_{\alpha,j} : 1 \leq j \leq \beta - 1\},$$

viewed as an ordered sequence of vertices from $z_\rho$ to $z_{P+1}$. It can be that some $z \in W$ with $z \neq z_{P+1}$ satisfies $z \in \partial F'$. If that holds, we find such $z$ with greatest suffix and remove all elements of $W$ with lesser suffix than $z$. Note, in this case, that $z \notin \{y_N, v, y_{N+1}, z_{P+1}\}$. See Figure 7.6.

This yields a doubly infinite path $\widehat{\nu}_1 = (\widehat{\nu}'_\rho(u-), W', \phi(F'), v, \widehat{\nu}(w-))$ of $\widehat{G}_\Delta$ where $W'$ is obtained from $W$ by $\phi$-removal and oxbow removal. We claim that $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$. To check this, it suffices to verify that there exist no $x \in (\widehat{\nu}'_\rho(u-), W')$ and $y \in \widehat{\nu}(w+)$ such that $x \mathrel{\widehat{\sim}} y$. This follows from Lemma 7.2(d) and a consideration based on whether or not $y_\alpha$ is a facial site.

Since $\widehat{\nu}_1$ includes the facial site $\phi(F')$, there exists $\overline{\nu} = \sigma^{-1}(\widehat{\nu}_1) \in \mathrm{NST}(G_*)$ that traverses a diagonal of $F'$, as required.

(B). *Suppose* $\lambda \leq P$. This is similar to Case (A).

(C). *Suppose either* $\rho = P$ *or* $\lambda = P+1$. Assume $\rho = P$; the other case is similar. By (7.10), we may add $\phi(F')$ to $\widehat{\nu}'_\rho(u-) \cup \{v\} \cup \widehat{\nu}(w+)$ to obtain the required $2\infty$-nst path $\widehat{\nu}_1$, and hence $\overline{\nu} = \sigma^{-1}(\widehat{\nu}_1)$ as before.

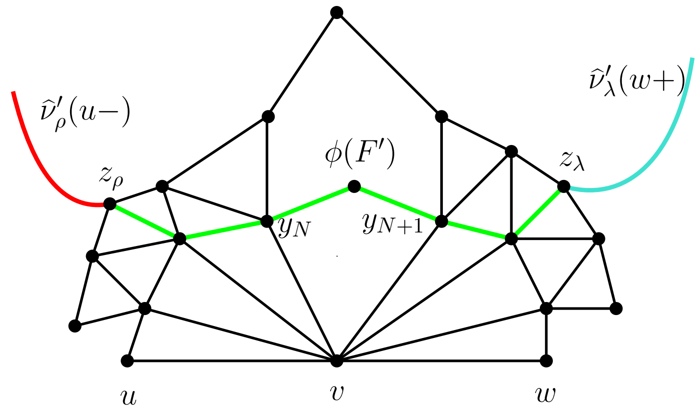


FIGURE 7.7. An illustration of $\widehat\nu_1$ in case (D.1), when the rightmost $U$ lies to the left and the leftmost $W$ lies to the right.

(D). *Suppose $\rho < P$ and $\lambda > P+1$.* Write $z_\rho = z_{\alpha,\beta}$ and $z_\lambda = z_{\gamma,\delta}$ (with $\alpha = 1$ if $\rho = 0$, and $\gamma = r$ if $\lambda = r+1$). There are two cases, depending on whether or not

(7.11)
$$\exists\, i,\, j \text{ with } \rho < i < P < P+1 < j < \lambda \text{ such that } z_i = z_j = \phi(J) \text{ for some } J.$$

1. Assume (7.11) does not hold. There is no pair $y_k$, $y_l$ with $\alpha < k \le N$, $N+1 \le l < \gamma$ that lie in the same facial cycle of $\widehat G_\Delta$. In this case we remove $\widehat\nu''_\rho(u-)$ and $\widehat\nu''_\lambda(w+)$ and add the vertices $y_\alpha, y_{\alpha+1}, \dots, y_N$, $\phi(F')$, and $y_{N+1}, y_{N+2}, \dots, y_\gamma$. The resulting set of vertices contains (after $\phi$-removal and oxbow removal) a $2\infty$-nst path $\widehat\nu_1 \in \sigma(\mathrm{NST}(G_*))$ that includes the facial site $\phi(F')$. The required $2\infty$-nst path of $G_*$ is $\overline\nu := \sigma^{-1}(\widehat\nu_1)$. See Figure 7.7.
2. Assume that (7.11) holds and pick $i$ least and then $j$ greatest. Write $z$ for the common vertex $z_i = z_j$ where $z = \phi(J)$ for some face $J$ of $\widehat G_\Delta$. It cannot be that both $\widehat\nu(u-) \cap \partial J \ne \varnothing$ and $\widehat\nu(w+) \cap \partial J \ne \varnothing$, since that contradicts $\nu \in \mathrm{NST}(G_*)$; assume then that $\widehat\nu(w+) \cap \partial J = \varnothing$. See Figure 7.8.
   (i) Suppose there exists $x \in \widehat\nu(u-) \cap \partial J$. By the planarity of $\widehat\nu$, it must be that $x \in \widehat\nu'_\rho(u-)$, and we pick such $x$ earliest with this property. We consider the walk
   $$(\widehat\nu(x-), z_i, z_{i+1} \dots, z_P, \phi(F'), v, \widehat\nu(w+)).$$
   After $\phi$-removal and oxbow removal, this becomes a $2\infty$-nst path $\widehat\nu_1$ of $\widehat G_\Delta$ lying in $\sigma(\mathrm{NST}(G_*))$. The required $2\infty$-nst path of $G_*$ is $\overline\nu := \sigma^{-1}(\widehat\nu_1)$.
   (ii) Suppose that $\widehat\nu(u-) \cap \partial J = \varnothing$. We apply the argument of the above case to the walk $(\widehat\nu'_\rho(u-), z_{\rho+1}, z_{\rho+2}, \dots, z_P, \phi(F'), v, \widehat\nu(w+))$.

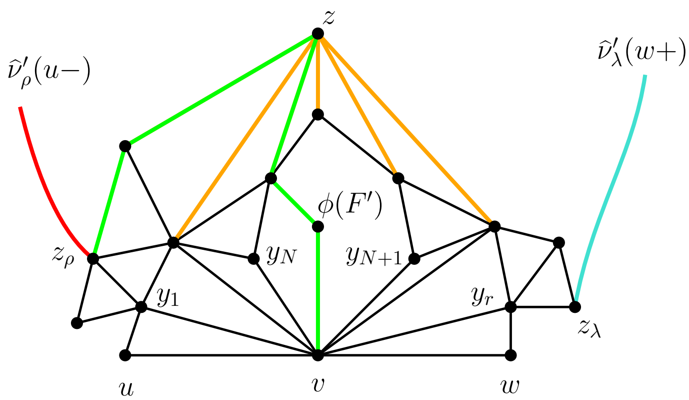


FIGURE 7.8. An illustration of $\widehat{\nu}_1$ in case (D.2)(ii). The vertex $z$ is a facial site in the face $J$, and is joined to $\partial J$ by the orange edges. The additional path from $z_\rho$ to $v$ is marked in green, and it makes use of the facial site $\phi(F')$.

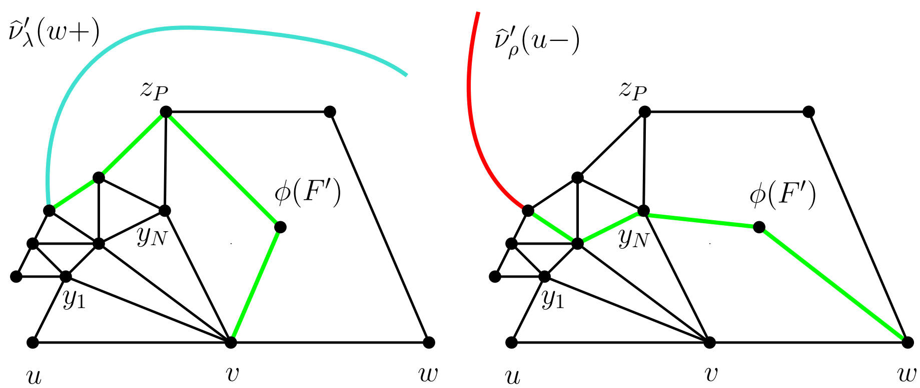


FIGURE 7.9. Illustrations of the constructions in Section 7.2. *Left*: When $\lambda \le P$, the path $\widehat{\nu}'_\lambda(w+)$ followed by certain vertices as marked results in a $2\infty$-nst path including the facial site $\phi(F')$. *Right*: When $\lambda \ge P+1$, the path $\widehat{\nu}'_\rho(u-)$ followed by certain vertices as marked forms a $2\infty$-nst path including $\phi(F')$.

7.2. **Case II: Suppose $\partial F'$ contains $\langle v, w\rangle$ but not $\langle u, v\rangle$.** The argument is similar to that of Section 7.1, and we sketch it. Let $y_1, y_2, \dots, y_N$ be the vertices adjacent to $v$ above the triple $u, v, w$, as illustrated in Figure 7.9. Let the $z_{i,j}$ be as in the last section, and let $(z_i : 1 \le i \le P)$, $z_\rho$, and $z_\lambda$ be given as before.

(E). *Suppose some $z_{i,j}$ is labelled $W$.* We proceed as in (A), (B) above. Find the leftmost such vertex, say $z_\lambda$. We delete $\widehat{\nu}''_\lambda(w+)$ from $\widehat{\nu}$ and add the $z_{i,j}$ that lie

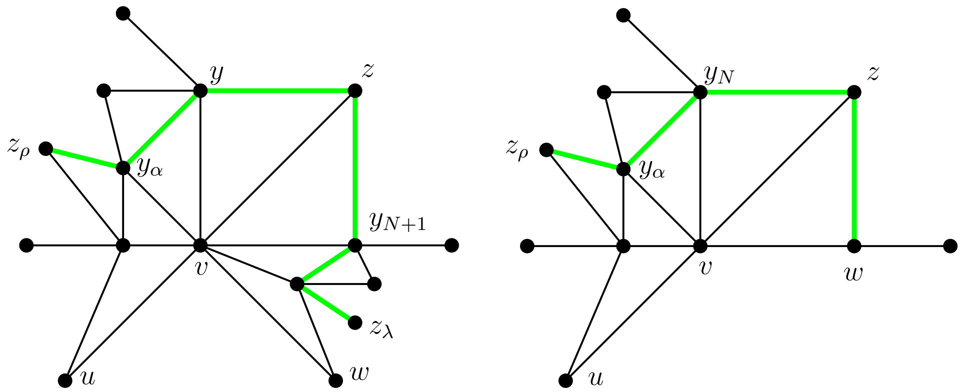


FIGURE 7.10. Illustrations of the constructions in Sections 8.1 and 8.2, respectively.

between $z_\lambda$ and $z_P$. This results (after $\phi$-removal and oxbow removal) in a $2\infty$-nst path $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ that includes the ordered sequence $(z_P, \phi(F'), \widehat{\nu}(u-))$.

(F). *Suppose no $z_{i,j}$ is labelled $W$.* We proceed as in (D) above. Find the rightmost $z_{i,j}$ labelled $U$, say $z_\rho = z_{\alpha,\beta}$ (with $\rho = 0$ if no such vertex exists). We delete $\widehat{\nu}''_\rho(u-)$ and $v$ from $\widehat{\nu}$ and add $y_\alpha, y_{\alpha+1} \dots, y_N$ to obtain a $2\infty$-nst path $\widehat{\nu}_1 \in \mathrm{NST}(G_*))$ that includes the ordered triple $(y_N, \phi(F'), w)$.

## 8. PROOF OF PROPOSITION 5.1(b)

Let $Q$ be a 4-cycle in $\widehat{G}_\Delta$ as in Figure 6.2, and note that some vertices of $Q$ may be facial sites. Let $\widehat{\nu} \in \sigma(\mathrm{NST}(G_*))$ be such that $v \in \widehat{\nu} \cap Q$. If $\widehat{\nu}$ includes some facial site, there is nothing more to prove, and so we may assume henceforth that

$$\widehat{\nu} \text{ includes no facial site.} \tag{8.1}$$

In particular, $u, v, w \in V$.

We may assume that $z \notin \widehat{\nu}$, since otherwise there is nothing to prove. In place of (7.1) we have (in the notation of Figure 6.2) that $\widehat{\nu} \cap Q$ is one of (i) the single vertex $v$, (ii) the single edge $\langle v, y' \rangle$, (iii) the single edge $\langle v, y \rangle$, (iv) the two edges $\langle v, y' \rangle$, $\langle v, y \rangle$. Case (iii) is handled as case (ii), and we proceed with cases (i), (ii), (iv) next.

8.1. **(i) Assume that $\widehat{\nu} \cap Q = \{v\}$,** and temporarily remove the edge $\langle v, z \rangle$ to obtain a 4-face $F$ with $\partial F = Q$. We shall reinstate this diagonal later.

We follow the constructions in the proof of Section 7.1. With the exception of case (D) of that section, we may take $\widehat{\nu}_1$ as given there (with the facial site $\phi(F')$ removed, so that the new path traverses the diagonal of $F$). Either $\widehat{\nu}_1$ includes some facial site or it does not, and in either case the claim follows.

We next consider case (D) with the diagonal reinstated in $F$, and see Figure 7.10. Vertices $z_\rho = z_{\alpha,\beta}$ and $z_\lambda = z_{\gamma,\delta}$ are as before. Since $\nu$ is non-self-touching and traverses no diagonal,

$$\widehat{\nu}'_\rho(u-) \not\widehat{\sim} \widehat{\nu}'_\lambda(w+). \tag{8.2}$$

1. If $z \not\widehat{\sim} \widehat{\nu}'_\rho(u-) \cup \widehat{\nu}'_\lambda(w+)$, we consider the walk
$$w = (\widehat{\nu}'_\rho(u-), y_\alpha, y_{\alpha+1}, \dots, y_N(=y), z, y_{N+1}(=y'), y_{N+2}, \dots, y_\gamma, \widehat{\nu}'_\lambda(w+)).$$
It may that $y_i \widehat{\sim} y_j$ for some $\alpha < i \le N$ and $N+1 \le j < \gamma$. This is treated as in case (D) of Section 7.1 (see (7.11)), which results (after $\rho$-removal and oxbow removal) in some $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ including $z$. Either $\widehat{\nu}_1$ includes some facial site or it does not, and in either case the claim is shown.
2. Assume $z \widehat{\sim} \widehat{\nu}'_\rho(u-)$ but $z \not\widehat{\sim} \widehat{\nu}'_\lambda(w+)$. Find the earliest $x \in \widehat{\nu}'_\rho(u-)$ satisfying $x \widehat{\sim} z$ (noting that $x \in V$); truncate $\widehat{\nu}'_\rho(u-)$ at $x$ to the subpath $\widehat{\nu}'(x-)$, and add the vertices $z, y_{N+1}, y_{N+2}, \dots, y_\gamma$ to $\widehat{\nu}'_\lambda(w-)$. Let $J$ be the face such that $z, x \in \overline{J}$; if $z \ne \phi(J)$ and $z \nsim x$ in $G$, we add $\phi(J)$ also. After $\rho$-removal and oxbow removal, one obtains the required $\widehat{\nu}_1$. It needs be checked that
$$\widehat{\nu}'(x-) \not\widehat{\sim} \{y_{N+1}, y_{N+2}, \dots, y_\gamma\}, \tag{8.3}$$
and this holds in a similar manner to that of case (A) of Section 7.1. A similar argument holds with $u$ and $w$ interchanged.
3. Assume $z \widehat{\sim} \widehat{\nu}'_\rho(u-)$ and $z \widehat{\sim} \widehat{\nu}'_\lambda(w+)$. By (8.2), $z \in V$. Find the earliest $x \in \widehat{\nu}'_\rho(u-)$ satisfying $x \widehat{\sim} z$, and the latest $y \in \widehat{\nu}'_\lambda(w+)$ satisfying $y \widehat{\sim} z$; truncate the two paths at $x$ and $y$ respectively, and add the vertex $z$ and any required facial site. The outcome is the required $\widehat{\nu}_1$.

8.2. **(ii) Assume that $\widehat{\nu} \cap Q$ is the edge $\langle v, w\rangle$,** where $w = y'$, and consider cases (E), (F) of Section 7.2. In (E), we may take $\widehat{\nu}_1$ to be as defined there. Consider the second case (F) as illustrated on the right of Figure 7.10. We follow Section 8.1 above but with differences as follows.

1. If $z \not\widehat{\sim} \widehat{\nu}'_\rho(u-) \cup \widehat{\nu}(w+) \setminus \{w\}$, we add to $\widehat{\nu}'_\rho(u-) \cup \widehat{\nu}(w+)$ the vertex sequence $y_\alpha, y_{\alpha+1}, \dots, y_N(=y), z$. If
$$\{y_\alpha, y_{\alpha+1}, \dots, y_N\} \not\widehat{\sim} \widehat{\nu}(w+), \tag{8.4}$$
the resulting path $\widehat{\nu}_1$ (after $\phi$-removal and oxbow removal) is as required. If (8.4) fails, we find the earliest $I$ such that $\alpha \le I \le N$ and $y_I \widehat{\sim} \widehat{\nu}(w+)$ and the latest $x \in \widehat{\nu}(w-)$ such that $y_I \widehat{\sim} x$. Note that $y_I, x \in V$, so that they lie in some common cycle $J$. Now apply $\phi$-removal and oxbow removal to the walk $(\widehat{\nu}_\rho(u-), y_\alpha, \dots, y_I, \phi(J), \widehat{\nu}(x-))$ to obtain $\widehat{\nu}_1 \in \sigma(\mathrm{NST}(G_*))$ that includes a facial site.

2. Assume $z \mathrel{\widehat{\sim}} \widehat{\nu}'_\rho(u-)$ but $z \mathrel{\widehat{\nsim}} \widehat{\nu}(w+) \setminus \{w\}$. Find the earliest $x \in \widehat{\nu}'_\rho(u-)$ satisfying $x \mathrel{\widehat{\sim}} z$ (noting that $x \in V$); truncate $\widehat{\nu}'_\rho(u-)$ at $x$, and add $z$ to $\widehat{\nu}(w+)$ (and also the facial site $\phi(J)$ if needed, as explained above), to obtain the required $\widehat{\nu}_1$. We argue similarly with $u$ and $w$ interchanged.
3. Assume $z \mathrel{\widehat{\sim}} \widehat{\nu}'_\rho(u-)$ and $z \mathrel{\widehat{\sim}} \widehat{\nu}(w+) \setminus \{w\}$. Find the earliest $x \in \widehat{\nu}'_\rho(u-)$ satisfying $x \mathrel{\widehat{\sim}} z$, and the latest $y \in \widehat{\nu}(w-)$ satisfying $y \mathrel{\widehat{\sim}} z$; truncate the two paths at $x$ and $y$ respectively, and add the vertex $z$ (possibly with facial sites as needed). The outcome is the required $\widehat{\nu}_1$.

8.3. **(iv) Assume that $\widehat{\nu} \cap Q$ comprises the two edges $\langle v, w\rangle$, $\langle v, y\rangle$,** so that $u = y$ and $w = y'$. The idea is to replace $v$ by $z$, and we proceed as above.

1. If $z \mathrel{\widehat{\nsim}} (\widehat{\nu}'(u-) \setminus \{u\}) \cup (\widehat{\nu}'(w+) \setminus \{w\})$, we remove $v$ from $\widehat{\nu}$ and add $z$ to $\widehat{\nu}'(u-) \cup \widehat{\nu}(w+)$.
2. Assume $z \mathrel{\widehat{\sim}} (\widehat{\nu}'(u-) \setminus \{u\})$ but $z \mathrel{\widehat{\nsim}} (\widehat{\nu}'(w+) \setminus \{w\})$. Find the earliest $x \in \widehat{\nu}'(u-)$ satisfying $x \mathrel{\widehat{\sim}} z$ (noting that $x \in V$); truncate $\widehat{\nu}'(u-)$ at $x$, and add $z$ to $\widehat{\nu}(w+)$ (and also the facial site $\phi(J)$ if needed, as explained above), to obtain the required $\widehat{\nu}_1$. We argue similarly with $u$ and $w$ interchanged.
3. Assume $z \mathrel{\widehat{\sim}} (\widehat{\nu}'(u-) \setminus \{u\})$ and $z \mathrel{\widehat{\sim}} (\widehat{\nu}'(w+) \setminus \{w\})$. Find the earliest $x \in \widehat{\nu}'(u-)$ satisfying $x \mathrel{\widehat{\sim}} z$, and the latest $y \in \widehat{\nu}'(w-)$ satisfying $y \mathrel{\widehat{\sim}} z$; truncate the two paths at $x$ and $y$ respectively, and add the vertex $z$ (possibly with facial sites as needed). The outcome is the required $\widehat{\nu}_1$.

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