

# Criticality, universality, and isoradiality

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**Abstract.** Critical points and singularities are encountered in the study of critical phenomena in probability and physics. We present recent results concerning the values of such critical points and the nature of the singularities for two prominent probabilistic models, namely percolation and the more general random-cluster model. The main topic is the statement and proof of the criticality and universality of the canonical measure of bond percolation on isoradial graphs (due to the author and Ioan Manolescu). The key technique used in this work is the star–triangle transformation, known also as the Yang–Baxter equation. The second topic reported here is the identification of the critical point of the random-cluster model on the square lattice (due to Beffara and Duminil-Copin), and of the criticality of the canonical measure of the random-cluster model with  $q \geq 4$  on periodic isoradial graphs (by the same authors with Smirnov). The proof of universality for percolation is expected to extend to the random-cluster model on isoradial graphs.

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## 1. Introduction

One of the most provocative and elusive problems in the mathematics of critical phenomena is the issue of *universality*. Disordered physical systems manifest phase transitions, the nature of which is believed to be independent of the local structure of space. Very little about universality is known rigorously for systems below their upper critical dimension. It is frequently said that “renormalization” is the key to universality, but rigorous applications of renormalization in the context of universality are rare.

There has been serious recent progress in the “exactly solvable” setting of the two-dimensional Ising model, and a handful of special cases for other models. Our principal purpose here is to outline recent progress concerning the identification of critical surfaces and the issue of universality for bond percolation and the random-cluster model on isoradial graphs, with emphasis on the general method, the current limitations, and the open problems.

For bond percolation on an extensive family of isoradial graphs, the canonical process, in which the star–triangle transformation is in harmony with the geometry, is shown to be critical. Furthermore, universality has been proved for this class of systems, at least for the critical exponents *at the critical surface*. These results, found in recent papers by the author and Manolescu, [27–29], vastly extend earlier calculations of critical values for the square lattice etc, with the added ingredient of universality. Note that, to date, we are able to prove

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only *conditional* universality: if a certain exponent exists for at least one isoradial graph, then a family of exponents exist for an extensive collection of isoradial graphs, and they are universal across this collection.

The picture for the general random-cluster model is more restrained, but significant progress has been achieved on the identification of critical points. The longstanding conjecture for the critical value of the square lattice has been proved by Beffara and Duminil-Copin [4], using a development of classical tools. Jointly with Smirnov [5], the same authors have used Smirnov's parafermionic observable in the first-order setting of  $q \geq 4$  to identify the critical surface of a periodic isoradial graph. It is conjectured that the methods of [29] may be extended to obtain universality for the random-cluster model on isoradial graphs.

The results reported in this survey are closely related to certain famous 'exact results' in the physics literature. Prominent in the latter regard is the book of Baxter [3], from whose preface we quote selectively as follows:

“... the phrase ‘exactly solved’ has been chosen with care. It is not necessarily the same as ‘rigorously solved’. ... There is of course still much to be done.”

Percolation is summarized in Section 2, and isoradial graphs in Section 3. Progress with criticality and universality for percolation are described in Section 4. Section 6 is devoted to recent progress with critical surfaces of random-cluster models on isoradial graphs, and open problems for percolation and the random-cluster model may be found in Sections 5 and 7.

## 2. Percolation

**2.1. Background.** Percolation is the fundamental stochastic model for spatial disorder. Since its introduction by Broadbent and Hammersley in 1957, it has emerged as a key topic in probability theory, with connections and impact across all areas of applied science in which disorder meets geometry. It is in addition a source of beautiful and apparently difficult mathematical problems, the solutions to which often require the development of new tools with broader applications.

Here is the percolation process in its basic form. Let  $G = (V, E)$  be an infinite, connected graph, typically a crystalline lattice such as the  $d$ -dimensional hypercubic lattice. We are provided with a coin that shows heads with some fixed probability  $p$ . For each edge  $e$  of  $G$ , we flip the coin, and we designate  $e$  *open* if heads shows, and *closed* otherwise. The open edges are considered open to the passage of material such as liquid, disease, or rumour.<sup>1</sup>

Liquid is supplied at a *source* vertex  $s$ , and it flows along the open edges and is blocked by the closed edges. The basic problem is to determine the geometrical properties (such as size, shape, and so on) of the region  $C_s$  that is wetted by the liquid. More generally, one is interested in the geometry of the connected subgraphs of  $G$  induced by the set of open edges. The components of this graph are called the *open clusters*.

Broadbent and Hammersley proved in [10, 30, 31] that there exists a *critical probability*  $p_c = p_c(G)$  such that: every open cluster is bounded if  $p < p_c$ , and some open cluster is unbounded if  $p > p_c$ . There are two *phases*: the *subcritical phase* when  $p < p_c$  and the

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<sup>1</sup>This is the process known as *bond* percolation. Later we shall refer to *site* percolation, in which the vertices (rather than the edges) receive random states.

*supercritical phase* when  $p > p_c$ . The singularity that occurs when  $p$  is near or equal to  $p_c$  has attracted a great deal of attention from mathematicians and physicists, and many of the principal problems remain unsolved even after several decades of study. See [22, 25] for general accounts of the theory of percolation.

Percolation is one of a large family of models of classical and quantum statistical physics that manifest phase transitions, and its theory is near the heart of the extensive scientific project to understand phase transitions and critical phenomena. Key aspects of its special position in the general theory include: (i) its deceptively simple formulation as a probabilistic model, (ii) its use as a comparator for more complicated systems, and (iii) its role in the development of new methodology.

One concrete connection between percolation and models for ferromagnetism is its membership of the one-parameter family of so-called random-cluster models. That is, percolation is the  $q = 1$  random-cluster model. The  $q = 2$  random-cluster model corresponds to the Ising model, and the  $q = 3, 4, \dots$  random-cluster models to the  $q$ -state Potts models. The  $q \downarrow 0$  limit is connected to electrical networks, uniform spanning trees, and uniform connected subgraphs. The *geometry* of the random-cluster model corresponds to the *correlation* structure of the Ising/Potts models, and thus its critical point  $p_c$  may be expressed in terms of the critical temperature of the latter systems. See [23, 64] for a general account of the random-cluster model.

The theory of percolation is extensive and influential. Not only is percolation a benchmark model for studying random spatial processes in general, but also it has been, and continues to be, a source of intriguing and beautiful open problems. Percolation in two dimensions has been especially prominent in the last decade by virtue of its connections to conformal invariance and conformal field theory. Interested readers are referred to the papers [14, 26, 54, 56, 57, 61, 63] and the books [6, 22, 25].

**2.2. Formalities.** For  $x, y \in V$ , we write  $x \leftrightarrow y$  if there exists an open path joining  $x$  and  $y$ . The *open cluster* at the vertex  $x$  is the set  $C_x = \{y : x \leftrightarrow y\}$  of all vertices reached along open paths from  $x$ , and we write  $C = C_0$  where 0 is a fixed vertex called the *origin*. Write  $\mathbb{P}_p$  for the relevant product probability measure, and  $\mathbb{E}_p$  for expectation with respect to  $\mathbb{P}_p$ .

The *percolation probability* is the function  $\theta(p)$  given by

$$\theta(p) = \mathbb{P}_p(|C| = \infty),$$

and the *critical probability* is defined by

$$p_c = p_c(G) = \sup\{p : \theta(p) = 0\}. \tag{2.1}$$

It is elementary that  $\theta$  is a non-decreasing function, and therefore,

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases}$$

It is not hard to see, by the Harris–FKG inequality, that the value  $p_c(G)$  does not depend on the choice of origin.

Let  $d \geq 2$ , and let  $\mathcal{L}$  be a  $d$ -dimensional lattice. It is a fundamental fact that  $0 < p_c(\mathcal{L}) < 1$ , but it is unproven in general that no infinite open cluster exists when  $p = p_c$

**Conjecture 2.1.** *For any lattice  $\mathcal{L}$  in  $d \geq 2$  dimensions, we have that  $\theta(p_c) = 0$ .*

The claim of the conjecture is known to be valid for certain lattices when  $d = 2$  and for large  $d$ , currently  $d \geq 15$ . This conjecture has been the ‘next open problem’ since the intensive study of the late 1980s.

Whereas the above process is defined in terms of a single parameter  $p$ , we are concerned here with the richer multi-parameter setting in which an edge  $e$  is designated open with some probability  $p_e$ . In such a case, the critical probability  $p_c$  is replaced by a so-called ‘critical surface’.

**2.3. Critical exponents and universality.** A great deal of effort has been directed towards understanding the nature of the percolation phase transition. The picture is now fairly clear for one specific model in two dimensions (site percolation on the triangular lattice), owing to the very significant progress in recent years linking critical percolation to the Schramm–Löwner curve  $\text{SLE}_6$ . There remain however substantial difficulties to be overcome even when  $d = 2$ , associated largely with the extension of such results to general two-dimensional systems. The case of large  $d$  (currently,  $d \geq 15$ ) is also well understood, through work based on the so-called ‘lace expansion’ (see [1]). Many problems remain open in the prominent case  $d = 3$ .

Let  $\mathcal{L}$  be a  $d$ -dimensional lattice. The nature of the percolation singularity on  $\mathcal{L}$  is expected to share general features with phase transitions of other models of statistical mechanics. These features are sometimes referred to as ‘scaling theory’ and they relate to the ‘critical exponents’ occurring in the power-law singularities (see [22, Chap. 9]). There are two sets of critical exponents, arising firstly in the limit as  $p \rightarrow p_c$ , and secondly in the limit over increasing spatial scales when  $p = p_c$ . The definitions of the critical exponents are found in Table 2.1 (taken from [22]).

The notation of Table 2.1 is as follows. We write  $f(x) \approx g(x)$  as  $x \rightarrow x_0 \in [0, \infty]$  if  $\log f(x)/\log g(x) \rightarrow 1$ . The *radius* of the open cluster  $C$  at the origin  $x$  is defined by

$$\text{rad}(C) = \sup\{\|y\| : x \leftrightarrow y\},$$

where

$$\|y\| = \sup_i |y_i|, \quad y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d,$$

is the supremum ( $L^\infty$ ) norm on  $\mathbb{R}^d$ . The limit as  $p \rightarrow p_c$  should be interpreted in a manner appropriate for the function in question (for example, as  $p \downarrow p_c$  for  $\theta(p)$ , but as  $p \rightarrow p_c$  for  $\kappa(p)$ ). The *indicator function* of an event  $A$  is denoted  $1_A$ .

Eight critical exponents are listed in Table 2.1, denoted  $\alpha, \beta, \gamma, \delta, \nu, \eta, \rho, \Delta$ , but there is no general proof of the existence of any of these exponents for arbitrary  $d \geq 2$ . Such critical exponents may be defined for phase transitions in a large family of physical systems. The exponents are not believed to be independent variables, but rather to satisfy the so-called *scaling relations*

$$\begin{aligned} 2 - \alpha &= \gamma + 2\beta = \beta(\delta + 1), \\ \Delta &= \delta\beta, \quad \gamma = \nu(2 - \eta), \end{aligned}$$

and, when  $d$  is not too large, the *hyperscaling relations*

$$d\rho = \delta + 1, \quad 2 - \alpha = d\nu.$$

More generally, a ‘scaling relation’ is any equation involving critical exponents which is believed to be ‘universally’ valid. The *upper critical dimension* is the largest value  $d_c$  such

<i>Function</i>		<i>Behaviour</i>	<i>Exp.</i>
percolation probability	$\theta(p) = \mathbb{P}_p( C  = \infty)$	$\theta(p) \approx (p - p_c)^\beta$	$\beta$
truncated mean cluster-size	$\chi^f(p) = \mathbb{E}_p( C 1_{ C <\infty})$	$\chi^f(p) \approx  p - p_c ^{-\gamma}$	$\gamma$
number of clusters per vertex	$\kappa(p) = \mathbb{E}_p( C ^{-1})$	$\kappa'''(p) \approx  p - p_c ^{-1-\alpha}$	$\alpha$
cluster moments	$\chi_k^f(p) = \mathbb{E}_p( C ^k 1_{ C <\infty})$	$\frac{\chi_{k+1}^f(p)}{\chi_k^f(p)} \approx  p - p_c ^{-\Delta}$	$\Delta$
correlation length	$\xi(p)$	$\xi(p) \approx  p - p_c ^{-\nu}$	$\nu$
cluster volume		$\mathbb{P}_{p_c}( C  = n) \approx n^{-1-1/\delta}$	$\delta$
cluster radius		$\mathbb{P}_{p_c}(\text{rad}(C) = n) \approx n^{-1-1/\rho}$	$\rho$
connectivity function		$\mathbb{P}_{p_c}(0 \leftrightarrow x) \approx \ x\ ^{2-d-\eta}$	$\eta$

Table 2.1. Eight functions and their critical exponents. The first five exponents arise in the limit as  $p \rightarrow p_c$ , and the remaining three as  $n \rightarrow \infty$  with  $p = p_c$ . See [22, p. 127] for a definition of the correlation length  $\xi(p)$ .

that the hyperscaling relations hold for  $d \leq d_c$  and not otherwise. It is believed that  $d_c = 6$  for percolation. There is no general proof of the validity of the scaling and hyperscaling relations for percolation, although quite a lot is known when either  $d = 2$  or  $d$  is large. The case of large  $d$  is studied via the lace expansion, and this is expected to be valid for  $d > 6$ .

We note some further points in the context of percolation.

- (a) **Universality.** The numerical values of critical exponents are believed to depend only on the value of  $d$ , and to be independent of the choice of lattice, and of the type of percolation under study.
- (b) **Two dimensions.** When  $d = 2$ , it is believed that

$$\alpha = -\frac{2}{3}, \beta = \frac{5}{36}, \gamma = \frac{43}{18}, \delta = \frac{91}{5}, \dots$$

These values (other than  $\alpha$ ) have been proved (essentially only) in the special case of site percolation on the triangular lattice, see [45, 60].

- (c) **Large dimensions.** When  $d$  is sufficiently large (in fact,  $d \geq d_c$ ) it is believed that the critical exponents are the same as those for percolation on a tree (the ‘mean-field model’), namely  $\delta = 2, \gamma = 1, \nu = \frac{1}{2}, \rho = \frac{1}{2}$ , and so on. Using the first hyperscaling

relation, this is consistent with the contention that  $d_c = 6$ . Several such statements are known to hold for  $d \geq 15$ , see [20, 32, 33, 41].

Open challenges include the following:

1. prove the existence of critical exponents for general lattices,
2. prove some version of universality,
3. prove the scaling and hyperscaling relations in general dimensions,
4. calculate the critical exponents for general models in two dimensions,
5. prove the mean-field values of critical exponents when  $d \geq 6$ .

Progress towards these goals has been substantial but patchy. As noted above, for sufficiently large  $d$ , the lace expansion has enabled proofs of exact values for many exponents, for a restricted class of lattices. There has been remarkable progress in recent years when  $d = 2$ , inspired largely by work of Cardy [14] and Schramm [53], enacted by Smirnov [56], and confirmed by the programme pursued by Lawler, Schramm, Werner, Camia, Newman, Sheffield and others to understand SLE curves and conformal ensembles.

In this paper, we concentrate on recent progress concerning isoradial embeddings of planar graphs, and particularly the identification of their critical surfaces and the issue of universality.

### 3. Isoradial graphs

Let  $G$  be an infinite, planar graph embedded in  $\mathbb{R}^2$  in such a way that edges intersect only at vertices. For simplicity, we assume that the embedding has only bounded faces. The graph  $G$  is called *isoradial* if (i) every face has a circumcircle which passes through every vertex of the face, (ii) the centre of each circumcircle lies in the interior of the corresponding face, and (iii) all such circumcircles have the same radius. We may assume by re-scaling that the common radius is 1.

The family of isoradial graphs is in two-to-one correspondence with the family of tilings of the plane with rhombi of side-length 1, in the following sense. Consider a rhombic tiling of the plane, as in Figure 3.1. The tiling, when viewed as a graph, is bipartite with vertex-sets coloured red and white, say. Fix a colour and join any two vertices of that colour whenever they are the opposite vertices of a rhombus. The resulting graph  $G$  is isoradial. If the other colour is chosen, the resulting graph is the (isoradial) dual of  $G$ . This is illustrated in Figures 3.1 and 3.2. Conversely, given an isoradial graph  $G$ , the corresponding rhombic tiling is obtained by augmenting its vertex-set by the circumcentres of the faces, and each circumcentre is joined to the vertices of the enclosing face.

Isoradial graphs were introduced by Duffin [17], and are related to the so-called  $\mathcal{Z}$ -invariant graphs of Baxter [2]. They were named thus by Kenyon, whose expository paper [36] proposes the connection between percolation and isoradiality (and much more). Isoradial graphs have two important properties, the first of which is their connection to pre-holomorphic functions. This was discovered by Duffin, and is summarized by Smirnov [59] and developed further in the context of probability by Chelkak and Smirnov [15]. This property is key to the work on the random-cluster model on isoradial graphs reviewed in Section

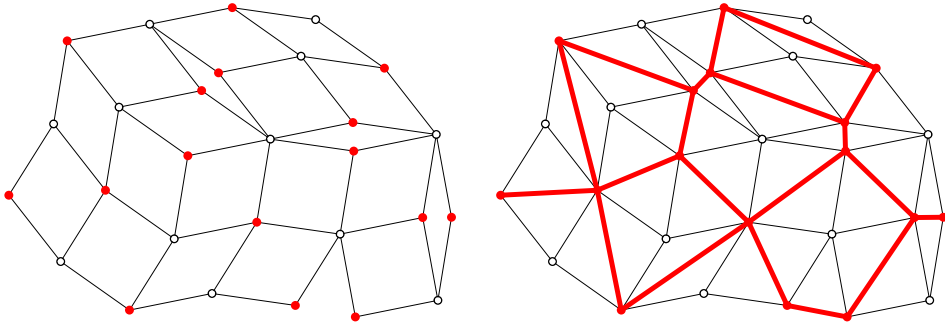


Figure 3.1. On the left is part of a rhombic tiling of the plane. Since all cycles have even length, this is a bipartite graph, with vertex-sets coloured red and white. The graph on the right is obtained by joining pairs of red vertices across faces. Each red face of the latter graph contains a unique white vertex, and this is the centre of the circumcircle of that face. Joining the white vertices, instead, yields another isoradial graph that is dual to the first.

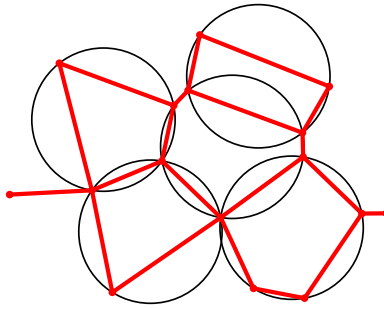


Figure 3.2. An illustration of the isoradiality of the red graph of Figure 3.1.

6. A recent review of connections between isoradiality and aspects of statistical mechanics may be found in [8].

The second property of isoradial graphs is of special relevance in the current work, namely that they provide the ‘right’ setting for the star–triangle transformation. This is explained next.

Consider an inhomogeneous bond percolation process on the isoradial graph  $G$ , whose edge-probabilities  $p_e$  are given as follows in terms of the graph-embedding. Each edge  $e$  of  $G$  is the diagonal of a unique rhombus in the corresponding rhombic tiling of the plane, and its parameter  $p_e$  is given in terms of the geometry of this rhombus. With  $\theta_e$  the opposite angle of the rhombus, as illustrated in Figure 3.3, let  $p_e \in (0, 1)$  satisfy

$$\frac{p_e}{1 - p_e} = \frac{\sin(\frac{1}{3}[\pi - \theta_e])}{\sin(\frac{1}{3}\theta_e)}. \tag{3.1}$$

We consider inhomogeneous bond percolation on  $G$  in which each edge  $e$  is designated open with probability  $p_e$ , and we refer to this as the *canonical percolation process* on  $G$ , with associated probability measure  $\mathbb{P}_G$ . The special property of the vector  $\mathbf{p} = (p_e : e \in E)$  is explained in Section 4.2.

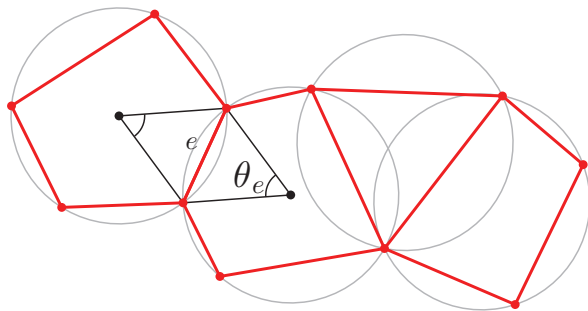


Figure 3.3. The edge  $e$  is the diagonal of some rhombus, with opposite angle  $\theta_e$  as illustrated.

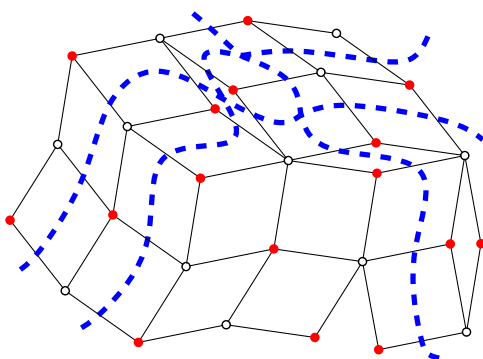


Figure 3.4. An illustration of the track system of the rhombic tiling of Figure 3.1.

In a beautiful series of papers [11–13], de Bruijn introduced the geometrical construct of ‘ribbons’ or ‘train tracks’ via which he was able to build a theory of rhombic tilings. Consider a tiling  $\mathbb{T}$  of the plane in which each tile is convex with four sides. We pursue a walk on the faces of  $\mathbb{T}$  according to the following rules. The walk starts in some given tile, and crosses some edge to a neighbouring tile. It next traverses the opposite edge of this tile, and so on. The walk may be extended backwards according to the same rule, and a doubly-infinite walk ensues. Such a walk is called a *ribbon* or *track*. De Bruijn pointed out that, if  $\mathbb{T}$  is a rhombic tiling, then no walk intersects itself, and two walks may intersect once but not twice. This property turns out to be both necessary and sufficient for a track system to be homeomorphic to that of a *rhombic* tiling (see [37]).

We impose two restrictions on the isoradial graphs under study. Firstly, we say that an isoradial graph  $G = (V, E)$  satisfies the *bounded-angles property* (BAP) if there exists  $\epsilon > 0$  such that

$$\epsilon < \theta_e < \pi - \epsilon \quad \text{for all } e \in E,$$

where  $\theta_e$  is as in Figure 3.3. This amounts to the condition that the rhombi in the corresponding tiling are not ‘too flat’. We say that  $G$  has the *square-grid property* (SGP) if its track system, viewed as a graph, contains a square grid such that those tracks not in the grid have boundedly many intersections with the grid within any bounded region (see [29, Sect. 4.2] for a more careful statement of this property).



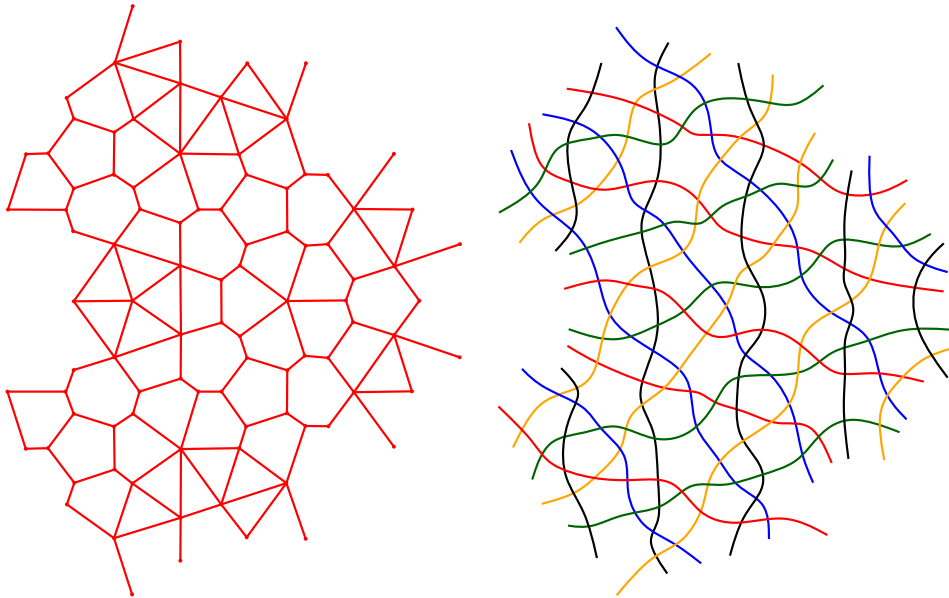


Figure 3.5. On the left, an isoradial graph obtained from part of the Penrose rhombic tiling. On the right, the associated track system comprises a pentagrid: five sets of non-intersecting doubly-infinite lines.

An isoradial graph may be viewed as both a graph and a planar embedding of a graph. Of the many examples of isoradial graphs, we mention first the conventional embeddings of the square, triangular, and hexagonal lattices. These are symmetric embeddings, and the edges have the same  $p$ -value. There are also non-symmetric isoradial embeddings of the same lattices, and indeed embeddings with no non-trivial symmetry, for which the corresponding percolation measures are ‘highly inhomogeneous’.

The isoradial family is much richer than the above examples might indicate, and includes graphs obtained from aperiodic tilings including the classic Penrose tiling [49, 50], illustrated in Figure 3.5. All isoradial graphs mentioned above satisfy the SGP, and also the BAP so long as the associated tiling comprises rhombi with flatness uniformly bounded from 0.

#### 4. Criticality and universality for percolation

**4.1. Two main theorems.** The first main theorem of [29] is the identification of the criticality of the canonical percolation measure  $\mathbb{P}_G$  on an isoradial graph  $G$ . The second is the universality of  $\mathbb{P}_G$  across an extensive family of isoradial graphs  $G$ .

In order to state the criticality theorem, we introduce notation that is appropriate for a perturbation of the canonical measure  $\mathbb{P}_G$ , and we borrow that of [5]. For  $e \in E$  and  $\beta \in (0, \infty)$ , let  $p_e(\beta)$  satisfy

$$\frac{p_e(\beta)}{1 - p_e(\beta)} = \beta \frac{\sin(\frac{1}{3}[\pi - \theta_e])}{\sin(\frac{1}{3}\theta_e)}, \tag{4.1}$$

and write  $\mathbb{P}_{G,\beta}$  for the corresponding product measure on  $G$ . Thus  $\mathbb{P}_{G,1} = \mathbb{P}_G$ .

**Theorem 4.1** (Criticality [29]). *Let  $G = (V, E)$  be an isoradial graph with the bounded-angles property and the square-grid property. The canonical percolation measure  $\mathbb{P}_G$  is critical in that*

(a) *there exist  $a, b, c, d > 0$  such that*

$$ak^{-b} \leq \mathbb{P}_G(\text{rad}(C_v) \geq k) \leq ck^{-d}, \quad k \geq 1, \quad v \in V,$$

(b) *there exists,  $\mathbb{P}_G$ -a.s., no infinite open cluster,*

(c) *for  $\beta < 1$ , there exist  $f, g > 0$  such that*

$$\mathbb{P}_{G,\beta}(|C_v| \geq k) \leq fe^{-gk}, \quad k \geq 0, \quad v \in V,$$

(d) *for  $\beta > 1$ , there exists,  $\mathbb{P}_{G,\beta}$ -a.s., a unique infinite open cluster.*

This theorem includes as special cases a number of known results for homogeneous and inhomogeneous percolation on the square, triangular, and hexagonal lattices beginning with Kesten's theorem that  $p_c = \frac{1}{2}$  for the square lattice, see [38, 39, 65].

We turn now to the universality of critical exponents. Recall the exponents  $\rho$ ,  $\eta$ , and  $\delta$  of Table 2.1. The exponent  $\rho_{2j}$  is the so-called  $2j$  alternating-arm critical exponent, see [26, 29]. An exponent is said to be  $\mathcal{G}$ -invariant if its value is constant across the family  $\mathcal{G}$ .

**Theorem 4.2** (Universality [29]). *Let  $\mathcal{G}$  be the class of isoradial graphs with the bounded-angles property and the square-grid property.*

(a) *Let  $\pi \in \{\rho\} \cup \{\rho_{2j} : j \geq 1\}$ . If  $\pi$  exists for some  $G \in \mathcal{G}$ , then it is  $\mathcal{G}$ -invariant.*

(b) *If either  $\rho$  or  $\eta$  exists for some  $G \in \mathcal{G}$ , then  $\rho$ ,  $\eta$ ,  $\delta$  are  $\mathcal{G}$ -invariant and satisfy the scaling relations  $\eta\rho = 2$  and  $2\rho = \delta + 1$ .*

The theorem establishes universality for bond percolation on isoradial graphs, but restricted to the exponents  $\rho$ ,  $\eta$ ,  $\delta$  at the critical point. The method of proof does not seem to extend to the near-critical exponents  $\beta$ ,  $\gamma$ , etc (see Problem E of Section 5).

It is in fact 'known' that, for reasonable two-dimensional lattices,

$$\rho = \frac{48}{5}, \quad \eta = \frac{5}{24}, \quad \delta = \frac{91}{5}, \tag{4.2}$$

although these values (and more), long predicted in the physics literature, have been proved rigorously only for (essentially) site percolation on the triangular lattice. See Lawler, Schramm, Werner [45] and Smirnov and Werner [60]. Note that site percolation on the triangular lattice does not lie within the ambit of Theorems 4.1 and 4.2.

To summarize, there is currently no known proof of the existence of critical exponents for any graph belonging to  $\mathcal{G}$ . However, if certain exponents exist for *any* such graph, then they exist for all  $G$  and are  $\mathcal{G}$ -invariant. If one could establish a result such as in (4.2) for any such graph, then this result would be valid across the entire family  $\mathcal{G}$ .

The main ideas of the proofs of Theorems 4.1 and 4.2 are as follows. The first element is the so-called box-crossing property. Loosely speaking, this is the property that the probability of an open crossing of a box with given aspect-ratio is bounded away from 0, uniformly in the position, orientation, and size of the box. The box-crossing property was proved by

Russo [52] and Seymour/Welsh [55] for homogeneous percolation on the square lattice, using its properties of symmetry and self-duality. It may be shown using classical methods that the box-crossing property is a certificate of a critical or supercritical percolation model. It may be deduced that, if both the primal and dual models have the box-crossing property, then they are both critical.

The star–triangle transformation of the next section provides a method for transforming one isoradial graph into another. The key step in the proofs is to show that this transformation preserves the box-crossing property. It follows that any isoradial graph that can be obtained by a sequence of transformations from the square lattice has the box-crossing property, and is therefore critical. It is proved in [29] that this includes any isoradial graph with both the BAP and SGP.

**4.2. Star–triangle transformation.** The central fact that permits proofs of criticality and universality is that the star–triangle transformation has a geometric representation that acts locally on rhombic tilings. Consider three rhombi assembled to form a hexagon as in the upper left of Figure 4.1. The interior of the hexagon may be tiled by (three) rhombi in either of two ways, the other such tiling being drawn at the upper right of the figure. The switch from the first to the second has two effects: (i) the track system is altered as indicated there, with one track being moved over the intersection of the other two, and (ii) the triangle in the isoradial graph of the upper left is transformed into a star. These observations are graph-theoretic rather than model-specific. We next parametrize the system in such a way that the parameters mutate in the canonical way under the above transformation. That is, for a given probabilistic model, we seek a parametrization under which the geometrical switch induces the appropriate parametric change.

Here is the star–triangle transformation for percolation. Consider the triangle  $T = (V, E)$  and the star  $S = (V', E')$  as drawn in Figure 4.2. Let  $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1]^3$ , and suppose the edges in the figure are declared open with the stated probabilities. The two ensuing configurations induce two connectivity relations on the set  $\{A, B, C\}$  within  $S$  and  $T$ , respectively. It turns out that these two connectivity relations are equi-distributed so long as  $\kappa(\mathbf{p}) = 0$ , where

$$\kappa(\mathbf{p}) = p_0 + p_1 + p_2 - p_1 p_2 p_3 - 1. \quad (4.3)$$

The star–triangle transformation is used as follows. Suppose, in a graph  $G$ , one finds a triangle whose edge-probabilities satisfy (4.3). Then this triangle may be replaced by a star having the complementary probabilities of Figure 4.2 without altering the probabilities of any long-range connections in  $G$ . Similarly, stars may be transformed into triangles. One complicating feature of the transformation is the creation of a new vertex when passing from a triangle to a star (and its destruction when passing in the reverse direction).

The star–triangle transformation was discovered first in the context of electrical networks by Kennelly [35] in 1899, and it was adapted in 1944 by Onsager [48] to the Ising model in conjunction with Kramers–Wannier duality. It is a key element in the work of Baxter [2, 3] on exactly solvable models in statistical mechanics, where it has become known as the *Yang–Baxter equation* (see [51] for a history of its importance in physics). Sykes and Essam [62] used the star–triangle transformation to predict the critical surfaces of certain inhomogeneous (but periodic) bond percolation processes on the triangular and hexagonal lattices, and furthermore the star–triangle transformation is a tool in the study of the random-cluster model [23, Sect. 6.6], and the dimer model [7].

Let us now explore the operation of the star–triangle transformation in the context of the

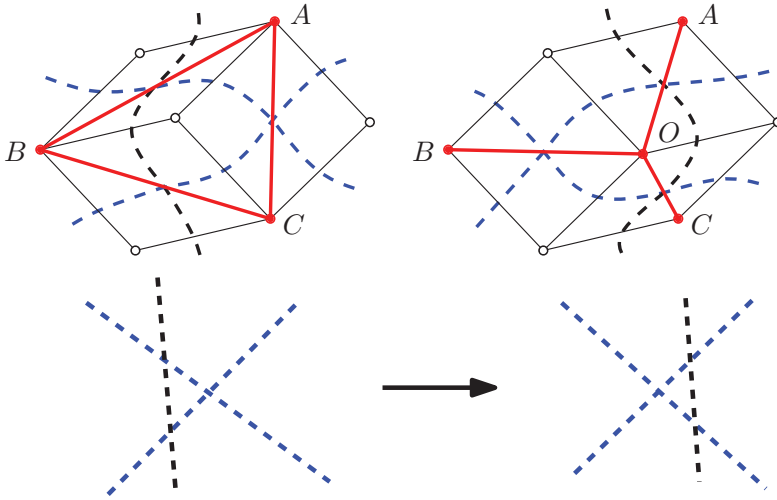


Figure 4.1. There are two ways of tiling the hexagon in the upper figure, and switching between these amounts to a star–triangle transformation for the isoradial graph. The effect on the track system is illustrated in the lower figure.

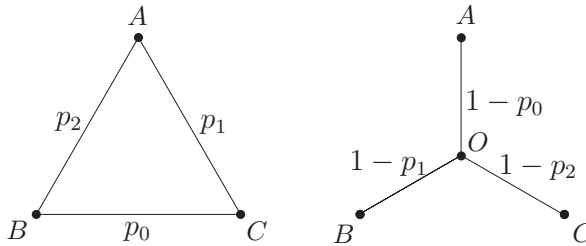


Figure 4.2. The star–triangle transformation for bond percolation.

rhombic switch of Figure 4.1. Let  $G$  be an isoradial graph containing the upper left hexagon of the figure, and let  $G'$  be the new graph after the rhombic switch. The definition (3.1) of the edge-probabilities has been chosen in such a way that the values on the triangle satisfy (4.3) and those on the star are as given in Figure 4.2. It follows that the connection probabilities on  $G$  and  $G'$  are equal. Graphs which have been thus parametrized but not embedded isoradially were called *Z-invariant* by Baxter [2]. See [44] for a recent account of the application of the above rhombic switch to Glauber dynamics of lozenge tilings of the triangular lattice.

One may couple the probability spaces on  $G$  and  $G'$  in such a way that the star–triangle transformation preserves *open connections*, rather than just their probabilities. Suppose that, in a given configuration, there exists an open path in  $G$  between vertex-sets  $A$  and  $B$ . On applying a sequence of star–triangle transformations, we obtain an open path in  $G'$  from the image of  $A$  to the image of  $B$ . Thus, star–triangle transformations transport open paths to open paths, and it is thus that the box-crossing property is transported from  $G$  to  $G'$ .

In practice, infinitely many star–triangle transformations are required to achieve the necessary transitions between graphs. The difficulties of the proofs of Theorems 4.1–4.2 are

centred on the need to establish sufficient control on the drifts of paths and their endvertices under these transformations.

## 5. Open problems for percolation

We discuss associated open problems in this section.

- (A) **Existence and equality of critical exponents.** It is proved in Theorem 4.2 that, if the three exponents  $\rho$ ,  $\eta$ ,  $\delta$  exist for some member of the family  $\mathcal{G}$ , then they exist for all members of the family, and are constant across the family. Essentially the only model for which existence has been proved is the site model on the triangular lattice, but this does not belong to  $\mathcal{G}$ . A proof of existence of exponents for the bond model on the square lattice would imply their existence for the isoradial graphs studied here. Similarly, if one can show any exact value for the latter bond model, then this value holds across  $\mathcal{G}$ .
- (B) **Cardy's formula.** Smirnov's proof [56] of Cardy's formula has resisted extension to models beyond site percolation on the triangular lattice. It seems likely that Cardy's formula is valid for canonical percolation on any reasonable isoradial graph. There is a strong sense in which the existence of interfaces is preserved under the star–triangle transformations of the proofs. On the other hand, there is currently only limited control of the geometrical perturbations of interfaces, and in addition Cardy's formula is as yet unproven for *all* isoradial bond percolation models.
- (C) **The bounded-angles property.** It is normal in working with probability and isoradial graphs to assume the BAP, see for example [15]. In the language of finite element methods, [9], the BAP is an example of the Ženíšek–Zlámal condition.

The BAP is a type of uniform non-flatness assumption. It implies an equivalence of metrics, and enables a uniform boundedness of certain probabilities. It may, however, not be *necessary* for the box-crossing property, and hence for the main results above.

As a test case, consider the situation in which all rhombi have angles exactly  $\epsilon$  and  $\pi - \epsilon$ . In the limit as  $\epsilon \downarrow 0$ , we obtain<sup>2</sup> the critical space–time percolation process on  $\mathbb{Z} \times \mathbb{R}$ , see Figure 5.1 and, for example, [24]. Let  $B_n(\alpha)$  be an  $n \times n$  square of  $\mathbb{R}^2$  inclined at angle  $\alpha$ , and let  $C_n(\alpha)$  be the event that the square is traversed by an open path between two given opposite faces. It is elementary using duality that

$$\mathbb{P}(C_n(\frac{1}{4}\pi)) \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \infty.$$

Numerical simulations (of A. Holroyd) suggest that the same limit holds when  $\alpha = 0$ . A proof of this would suggest that the limit does not depend on  $\alpha$ , and this in turn would support the possibility that the critical space–time percolation process satisfies Cardy's formula.

- (D) **The square-grid property.** The SGP is a useful tool in the proof of Theorem 4.2, but it may not be necessary. In [29] is presented an isoradial graph without the SGP, and this example may be handled using an additional ad hoc argument.

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<sup>2</sup>Joint work with Omer Angel.

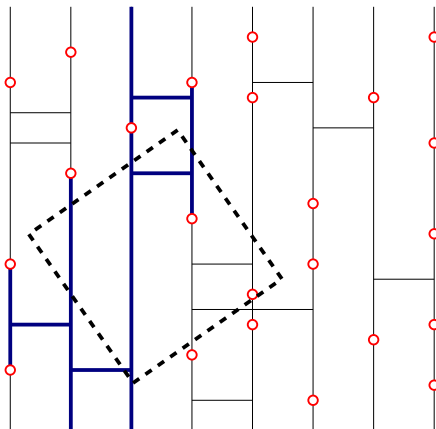


Figure 5.1. *Space–time percolation.* Each line is cut at rate 1, and nearest neighbours are joined at rate 1. One of the open clusters is highlighted. We ask for the probability that the box is traversed by an open path from its lower left side to its upper right side.

- (E) **Near-critical exponents.** Theorem 4.2 establishes the universality of exponents *at criticality*. The method of proof does not appear to be extendable to the *near-critical exponents*, and it is an open problem to prove these to be universal for isoradial graphs. Kesten showed in [40] (see also [47]) that certain properties of a critical percolation process imply properties of the near-critical process, so long as the underlying graph has a sufficiently rich automorphism group. In particular, for such graphs, knowledge of certain critical exponents at criticality implies knowledge of exponents away from criticality. Only certain special isoradial graphs have sufficient homogeneity for such arguments to hold without new ideas of substance, and it is an open problem to weaken these assumptions of homogeneity. See the discussion around [28, Thm 1.2].
- (F) **Random-cluster models.** How far may the proofs be extended to other models? It may seem at first sight that only a star–triangle transformation is required, but, as usual in such situations, boundary conditions play a significant role for dependent models such as the random-cluster model. The control of boundary conditions presents a new difficulty, so far unexplained. We return to this issue in Section 7.

## 6. Random-cluster model

**6.1. Background.** The random-cluster model was introduced by Fortuin and Kasteleyn around 1970 as a unification of processes satisfying versions of the series and parallel laws. In its base form, the random-cluster model has two parameters, an edge-parameter  $p$  and a cluster-weighting factor  $q$ .

Let  $G = (V, E)$  be a finite graph, with associated configuration space  $\Omega = \{0, 1\}^E$ . For  $\omega \in \Omega$  and  $e \in E$ , the edge  $e$  is designated *open* if  $\omega_e = 1$ . Let  $k(\omega)$  be the number of open clusters of a configuration  $\omega$ . The *random-cluster measure* on  $\Omega$ , with parameters  $p \in [0, 1]$ ,

$q \in (0, \infty)$ , is the probability measure satisfying

$$\phi_{p,q}(\omega) \propto q^{k(\omega)} \mathbb{P}_p(\omega), \quad \omega \in \Omega, \quad (6.1)$$

where  $\mathbb{P}_p$  is the percolation product-measure with density  $p$ . In a more general setting, each edge  $e \in E$  has an associated parameter  $p_e$ .

Bond percolation is retrieved by setting  $q = 1$ , and electrical networks arise via the limit  $p, q \rightarrow 0$  in such a way that  $q/p \rightarrow 0$ . The relationship to Ising/Potts models is more complicated and involves a transformation of measures. In brief, two-point connection probabilities for the random-cluster measure with  $q \in \{2, 3, \dots\}$  correspond to two-point correlations for ferromagnetic  $q$ -state Ising/Potts models, and this allows a geometrical interpretation of the latter's correlation structure. A fuller account of the random-cluster model and its history and associations may be found in [23, 64], to which the reader is referred for the basic properties of the model.

The special cases of percolation and the Ising model are very much better understood than is the general random-cluster model. We restrict ourselves to two-dimensional systems in this review, and we concentrate on the question of the identification of critical surfaces for certain isoradial graphs.

Two pieces of significant recent progress are reported here. Firstly, Beffara and Duminil-Copin [4] have developed the classical approach of percolation in order to identify the critical point of the square lattice, thereby solving a longstanding conjecture. Secondly, together with Smirnov [5], they have made use of the so-called parafermionic observable of [58] in a study of the critical surfaces of periodic isoradial graphs with  $q \geq 4$ .

**6.2. Formalities.** The random-cluster measure may not be defined directly on an *infinite* graph  $G$ . There are two possible ways to proceed in the setting of an infinite graph, namely via either boundary conditions or the DLR condition. The former approach works as follows. Let  $(G_n : n \geq 1)$  be an increasing family of finite subgraphs of  $G$  that exhaust  $G$  in the limit  $n \rightarrow \infty$ , and let  $\partial G_n$  be the *boundary* of  $G_n$ , that is,  $\partial G_n$  is the set of vertices of  $G_n$  that are adjacent to a vertex of  $G$  not in  $G_n$ . A *boundary condition* is an equivalence relation  $b_n$  on  $\partial G_n$ ; any two vertices  $u, v \in \partial G_n$  that are equivalent under  $b_n$  are taken to be part of the same cluster. The extremal boundary conditions are: the *free* boundary condition, denoted  $b_n = 0$ , for which each vertex is in a separate equivalence class; and the *wired* boundary condition, denoted  $b_n = 1$ , with a unique equivalence class. We now consider the set of weak limits as  $n \rightarrow \infty$  of the random-cluster measures on  $G_n$  with boundary conditions  $b_n$ .

Assume henceforth that  $q \geq 1$ . Then the random-cluster measures have properties of positive association and stochastic ordering, and one may deduce that the free and wired boundary conditions  $b_n = 0$  and  $b_n = 1$  are extremal in the following sense: (i) there is a unique weak limit of the free measures (respectively, the wired measures), and (ii) any other weak limit lies, in the sense of stochastic ordering, between these two limits. We write  $\phi_{p,q}^0$  and  $\phi_{p,q}^1$  for the free and wired weak limits. It is an important question to determine when  $\phi_{p,q}^0 = \phi_{p,q}^1$ , and the answer so far is incomplete even when  $G$  has a periodic structure, see [23, Sect. 5.3].

The *percolation probabilities* are defined by

$$\theta^b(p, q) = \phi_{p,q}^b(0 \leftrightarrow \infty), \quad b = 0, 1, \quad (6.2)$$

and the *critical values* by

$$p_c^b(q) = \sup\{p : \theta^b(p, q) = 0\}, \quad b = 0, 1. \quad (6.3)$$

Suppose that  $G$  is embedded in  $\mathbb{R}^d$  in a natural manner. When  $G$  is *periodic* (that is, its embedding is invariant under a  $\mathbb{Z}^d$  action), there is a general argument using convexity of pressure (see [21]) that implies that  $p_c^0(q) = p_c^1(q)$ , and in this case we write  $p_c(q)$  for the common value.

One of the principal problems is to determine for which  $q$  the percolation probability  $\theta^1(p, q)$  is discontinuous at the critical value  $p_c$ . This amounts to asking when  $\theta^1(p_c, q) > 0$ ; the phase transition is said to be of *first order* whenever the last inequality holds. The phase transition is known to be of first order for sufficiently large  $q$ , and is believed to be so if and only if  $q > Q(d)$  for some  $Q(d)$  depending on the dimension  $d$ . Furthermore, it is expected that

$$Q(d) = \begin{cases} 4 & \text{if } d = 2, \\ 2 & \text{if } d \geq 4. \end{cases}$$

We restrict our attention henceforth to the case  $d = 2$ , for which it is believed that the value  $q = 4$  separates the first and second order transitions. Recall Conjecture 2.1 and note the recent proof that  $Q(2) \geq 4$ , for which the reader is referred to [18] and the references therein.

**6.3. Critical point on the square lattice.** The square lattice  $\mathbb{Z}^2$  is one of the main playgrounds of physicists and probabilists. Although the critical points of percolation, the Ising model and some Potts models on  $\mathbb{Z}^2$  are long proved, the general answer for random-cluster models (and hence all Potts models) has been proved only recently.

**Theorem 6.1** (Criticality [4]). *The random-cluster model on the square lattice with cluster-weighting factor  $q \geq 1$  has critical value*

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

This exact value has been ‘known’ for a long time. When  $q = 1$ , the statement  $p_c(1) = \frac{1}{2}$  is the Harris–Kesten theorem for bond percolation. When  $q = 2$ , it amounts to the well known calculation of the critical temperature of the Ising model. For large  $q$ , the result (and more) was proved in [42, 43] ( $q > 25.72$  suffices, see [23, Sect. 6.4]). There is a ‘physics proof’ in [34] for  $q \geq 4$ .

The main contribution of [4] is a proof of a box-crossing property using a clever extension of the ‘RSW’ arguments of Russo and Seymour–Welsh in the context of the symmetry illustrated in Figure 6.1, combined with careful control of boundary conditions. An alternative approach is developed in [19].

**6.4. Isoradiality and the star–triangle transformation.** The star–triangle transformation for the random-cluster model is similar to that of percolation, and is illustrated in Figure 6.2. The three edges of the triangle have parameters  $p_0, p_1, p_2$ , and we set  $\mathbf{y} = (y_0, y_1, y_2)$  where

$$y_i = \frac{p_i}{1 - p_i}.$$

The corresponding edges of the star have parameters  $y'_i$  where  $y_i y'_i = q$ . Finally, we require that the  $y_i$  satisfy  $\psi(\mathbf{y}) = 0$  where

$$\psi(\mathbf{y}) = y_1 y_2 y_3 + y_1 y_2 + y_2 y_3 + y_3 y_1 - q. \quad (6.4)$$



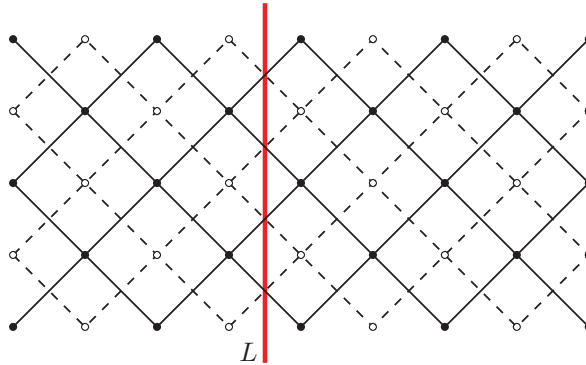


Figure 6.1. The square lattice and its dual, rotated through  $\pi/4$ . Under reflection in the line  $L$ , the primal is mapped to the dual.

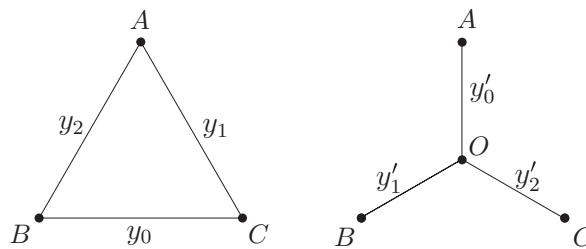


Figure 6.2. The star–triangle transformation for the random-cluster model.

Further details of the star–triangle transformation for the random-cluster model may be found in [23, Sect. 6.6].

We now follow the discussion of Section 4.2 of the relationship between the star–triangle transformation and the rhombus-switch of Figure 4.1. In so doing, we arrive (roughly as in [36, p. 282]) at the ‘right’ parametrization for an isoradial graph  $G$ , namely with (3.1) replaced by

$$\begin{aligned}
 \text{if } 1 \leq q < 4: \quad y_e &= \sqrt{q} \frac{\sin(\frac{1}{2}\sigma(\pi - \theta_e))}{\sin(\frac{1}{2}\sigma\theta_e)}, & \cos(\frac{1}{2}\sigma\pi) &= \frac{1}{2}\sqrt{q}, \\
 \text{if } q > 4: \quad y_e &= \sqrt{q} \frac{\sinh(\frac{1}{2}\sigma(\pi - \theta_e))}{\sinh(\frac{1}{2}\sigma\theta_e)}, & \cosh(\frac{1}{2}\sigma\pi) &= \frac{1}{2}\sqrt{q},
 \end{aligned}
 \tag{6.5}$$

where  $\theta_e$  is given in Figure 3.3. The intermediate case  $q = 4$  is the common limit of the two expressions as  $q \rightarrow 4$ , namely

$$y_e = 2 \frac{\pi - \theta_e}{\theta_e}.$$

Write  $\phi_{G,q}^b$  for the corresponding random-cluster measure on an isoradial graph  $G$  with boundary condition  $b = 0, 1$ . We refer to  $\phi_{G,q}^0$  as the ‘canonical random-cluster measure’ on  $G$ .

**6.5. Criticality via the parafermion.** Theorem 6.1 is proved in [4] by classical methods, and it holds for all  $q \geq 1$ . The proof is sensitive to the assumed symmetries of the lattice, and does not currently extend even to the inhomogeneous random-cluster model on  $\mathbb{Z}^2$  in which the vertical and horizontal edges have different parameter values. In contrast, the parafermionic observable introduced by Smirnov [58] has been exploited by Beffara, Duminil-Copin, and Smirnov [5] to study the critical point of fairly general isoradial graphs subject to the condition  $q \geq 4$ .

Let  $G = (V, E)$  be an isoradial graph. For  $\beta \in (0, \infty)$ , let  $y_e(\beta) = \beta y_e$  where  $y_e$  is given in (6.5). Let

$$p_e(\beta) = \frac{y_e(\beta)}{1 + y_e(\beta)}$$

accordingly, and write  $\phi_{G,q,\beta}^b$  for the corresponding random-cluster measure on  $G$  with boundary condition  $b$ . The following result of [5] is proved by a consideration of the parafermionic observable.

**Theorem 6.2** ([5]). *Let  $q \geq 4$ , and let  $G$  be an isoradial graph satisfying the BAP. For  $\beta < 1$ , there exists a  $a > 0$  such that*

$$\phi_{G,q,\beta}^0(u \leftrightarrow v) \leq e^{-a|u-v|}, \quad u, v \in V.$$

One deduces from Theorem 6.2 using duality that

- (a) for  $\beta < 1$ ,  $\phi_{G,q,\beta}^0$ -a.s., there is no infinite open cluster, and
- (b) for  $\beta > 1$ ,  $\phi_{G,q,\beta}^1$ -a.s., there exists a unique infinite open cluster.

This is only a partial verification of the criticality of the canonical measure, since parts (a) and (b) deal with potentially different measures, namely the free and wired limit measures, respectively. Further progress may be made for *periodic* graphs, as follows. Subject to the assumption of periodicity, it may be proved as in [21] that  $\phi_{G,q,\beta}^0 = \phi_{G,q,\beta}^1$  for almost every  $\beta$ , and hence that part (b) holds with  $\phi_{G,q,\beta}^1$  replaced by  $\phi_{G,q,\beta}^0$ . Therefore, for periodic embeddings, the canonical measure  $\phi_{G,q}^0 = \phi_{G,q,1}^0$  is critical.

Here is an application of the above remarks to the (periodic) inhomogeneous square lattice.

**Corollary 6.3** ([5]). *Let  $q \geq 4$ , and consider the random-cluster model on  $\mathbb{Z}^2$  with the variation that horizontal edges have parameter  $p_1$  and vertical edges parameter  $p_2$ . The critical surface is given by  $y_1 y_2 = q$  where  $y_i = p_i / (1 - p_i)$ .*

We close with the observation that a great deal more is known in the special case when  $q = 2$ . The  $q = 2$  random-cluster model corresponds to the Ising model, for which the special arithmetic of the equation  $1 + 1 = 2$  permits a number of techniques which are not available in greater generality. In particular, the Ising model and the  $q = 2$  random-cluster model on an isoradial graph lend themselves to a fairly complete theory using the parafermionic observable. The interested reader is directed to the work of Smirnov [57, 58] and Chelkak–Smirnov [16].

## 7. Open problems for the random-cluster model

- (A) **Inhomogeneous models.** Extend Corollary 6.3 to cover the case  $1 \leq q < 4$ .

- (B) **Periodicity.** Remove the assumption of periodicity in the proof of criticality of the canonical random-cluster measure on isoradial graphs. It would suffice to prove that  $\phi_{G,q,\beta}^0 = \phi_{G,q,\beta}^1$  for almost every  $\beta$ , without the assumption of periodicity. More generally, it would be useful to have a proof of the uniqueness of Gibbs states for *aperiodic* interacting systems, along the lines of that of Lebowitz and Martin-Löf [46] for a *periodic* Ising model.
- (C) **Bounded-angles property.** Remove the assumption of the bounded-angles property in Theorem 6.1.
- (D) **Criticality and universality for general  $q$ .** Adapt the arguments of [29] (or otherwise) to prove criticality and universality for the canonical random-cluster measure on isoradial graphs either for general  $q \geq 1$  or subject to the restriction  $q \geq 4$ .

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