Preface to the Fourth Edition

This book provides an extensive introduction to probability and random processes. It is intended for those working in the many and varied applications of the subject as well as for those studying more theoretical aspects. We hope it will be found suitable for mathematics undergraduates at all levels, as well as for graduate students and others with interests in these fields.

In particular, we aim:

- to give a rigorous introduction to probability theory while limiting the amount of measure theory in the early chapters;
- to discuss the most important random processes in some depth, with many examples;
- to include various topics which are suitable for undergraduate courses, but are not routinely taught;
- to impart to the beginner the flavour of more advanced work, thereby whetting the appetite for more.

The ordering and numbering of material in this fourth edition has for the most part been preserved from the third. However, a good many minor alterations and additions have been made in the pursuit of clearer exposition. Furthermore, we have revised extensively the sections on Markov chains in continuous time, and added new sections on coupling from the past, Lévy processes, self-similarity and stability, and time changes.

In an immoderate manifestation of millennial mania, the number of exercises and problems has been increased beyond the statutory 1000 to a total of 1322. Moreover, many of the existing exercises have been refreshed by additional parts, making a total of more than 3000 challenges for the diligent reader. These are frequently far from being merely drill exercises, but they complement and illustrate the text, or are entertaining, or (usually, we hope) both. In a companion volume, *One Thousand Exercises in Probability* (Oxford University Press, third edition, 2020), we give worked solutions to almost all exercises and problems.

The basic layout of the book remains unchanged. Chapters 1–5 begin with the foundations of probability theory, move through the elementary properties of random variables, and finish with the weak law of large numbers and the central limit theorem; on route, the reader meets random walks, branching processes, and characteristic functions. This material is suitable for about two lecture courses at a moderately elementary level.

The second part of the book is devoted to the theory of random processes. Chapter 6 deals with Markov chains in discrete and continuous time. The treatment of discrete-time chains is quite detailed and includes an easy proof of the limit theorem for chains with countably infinite state spaces. The sections on continuous-time chains provide a significant amplification of those of the third edition, and constitute an approachable but rigorous account of the principal theory and applications. Chapter 7 contains a general discussion of convergence, together with simple, rigorous accounts of the strong law of large numbers, and of martingale convergence. Each of these two chapters could be used as a basis for a lecture course.

Chapter 8 is an introduction to stochastic processes in their various types; most of these are studied in detail in later chapters, but we have also aspired there to engage the reader with wider aspects of probability by including new essays on a number of topics such as Lévy processes, self-similarity, and stability. Chapters 8–13 provide suitable material for about five shorter lecture courses on: stationary processes and ergodic theory; renewal processes; queues; martingales; diffusions and stochastic integration with applications to finance.

We thank those who have read and commented upon sections of this and earlier editions, and also those readers who have taken the trouble to write to us with notice of imperfections. Further help in thinning any remaining errors will be greatly appreciated.

Cambridge and Oxford G.R.G.
April 2020 D.R.S.

Note on the Frontispiece

The iconography associated with Fortuna, the goddess of chance and luck, (Tyche or Agathodaemon to the Greeks), has accumulated over more than two millennia, and is correspondingly complex; we give no more than a brief and much simplified account here of the various allegories involved. The goddess Fortuna was originally associated with fertility, hence the sheaf of corn shown in her right hand. Later associations with the uncertainty of sea voyages are indicated by the ship in full sail seen in the background. The sphere by her feet may initially have represented the instability of life, and this interpretation is sometimes strengthened by depicting the sphere as a bubble. (By contrast, the goddess Virtue is frequently depicted on or by a cube, representing stability.) Subsequently the sphere comes to represent the entire world, over which chance reigns supreme. The wheel carried by Fortuna in her left hand represents the fickleness and uncertainty entailed by the passage of time; that is, the inevitable ups and downs of life. 'The wheel of fortune' is a metaphor for chance and uncertainty still today, as exemplified by the title of a recent television game show.

A further discussion of the iconography is provided in Roberts 1998.

Contents

1 Events and their probabilities

1.1 Introduction 1
1.2 Events as sets 1
1.3 Probability 4
1.4 Conditional probability 8
1.5 Independence 13
1.6 Completeness and product spaces 15
1.7 Worked examples 16
1.8 Problems 22

2 Random variables and their distributions

2.1 Random variables 28
2.2 The law of averages 32
2.3 Discrete and continuous variables 35
2.4 Worked examples 37
2.5 Random vectors 40
2.6 Monte Carlo simulation 43
2.7 Problems 45

3 Discrete random variables

3.1 Probability mass functions 49
3.2 Independence 51
3.3 Expectation 53
3.4 Indicators and matching 59
3.5 Examples of discrete variables 64
3.6 Dependence 66
3.7 Conditional distributions and conditional expectation 71
3.8 Sums of random variables 75
3.9 Simple random walk 77
3.10 Random walk: counting sample paths 81
3.11 Problems 89

4 Continuous random variables

4.1 Probability density functions 99
4.2 Independence 101
4.3 Expectation 103
4.4 Examples of continuous variables 106
4.5 Dependence 110
4.6 Conditional distributions and conditional expectation 117
4.7 Functions of random variables 121
4.8 Sums of random variables 129
4.9 Multivariate normal distribution 131
4.10 Distributions arising from the normal distribution 135
4.11 Sampling from a distribution 138
4.12 Coupling and Poisson approximation 144
4.13 Geometrical probability 150
4.14 Problems 157

5 Generating functions and their applications

5.1 Generating functions 168
5.2 Some applications 176
5.3 Random walk 183
5.4 Branching processes 192
5.5 Age-dependent branching processes 197
5.6 Expectation revisited 199
5.7 Characteristic functions 203
5.8 Examples of characteristic functions 208
5.9 Inversion and continuity theorems 212
5.10 Two limit theorems 216
5.11 Large deviations 226
5.12 Problems 231

6 Markov chains

6.1 Markov processes 239
6.2 Classification of states 246
6.3 Classification of chains 250
6.4 Stationary distributions and the limit theorem 254
6.5 Reversibility 265
6.6 Chains with finitely many states 269
6.7 Branching processes revisited 273
6.8 Birth processes and the Poisson process 276
6.9 Continuous-time Markov chains 288
6.10 Kolmogorov equations and the limit theorem 297
6.11 Birth–death processes and imbedding 306
6.12 Special processes 313
6.13 Spatial Poisson processes 319
6.14 Markov chain Monte Carlo 330
6.15 Problems 339

7 Convergence of random variables

7.1 Introduction 349
7.2 Modes of convergence 352
7.3 Some ancillary results 362
7.4 Laws of large numbers 370
7.5 The strong law 374
7.6 The law of the iterated logarithm 377
7.7 Martingales 378
7.8 Martingale convergence theorem 383
7.9 Prediction and conditional expectation 388
7.10 Uniform integrability 395
7.11 Problems 400

8 Random processes

8.1 Introduction 406
8.2 Stationary processes 407
8.3 Renewal processes 411
8.4 Queues 414
8.5 The Wiener process 416
8.6 Lévy processes and subordinators 418
8.7 Self-similarity and stability 421
8.8 Time changes 426
8.9 Existence of processes 429
8.10 Problems 431

9 Stationary processes

9.1 Introduction 433
9.2 Linear prediction 436
9.3 Autocovariances and spectra 438
9.4 Stochastic integration and the spectral representation 446
9.5 The ergodic theorem 452
9.6 Gaussian processes 464
9.7 Problems 468

10 Renewals

10.1 The renewal equation 471
10.2 Limit theorems 476
10.3 Excess life 480
10.4 Applications 483
10.5 Renewal–reward processes 491
10.6 Problems 497

11 Queues

11.1 Single-server queues 501
11.2 M/M/1 503
11.3 M/G/1 506

11.4 G/M/1 512
11.5 G/G/1 516
11.6 Heavy traffic 523
11.7 Networks of queues 524
11.8 Problems 530

12 Martingales

12.1 Introduction 533
12.2 Martingale differences and Hoeffding's inequality 538
12.3 Crossings and convergence 544
12.4 Stopping times 550
12.5 Optional stopping 554
12.6 The maximal inequality 559
12.7 Backward martingales and continuous-time martingales 562
12.8 Some examples 567
12.9 Problems 572

13 Diffusion processes

13.1 Introduction 577
13.2 Brownian motion 578
13.3 Diffusion processes 580
13.4 First passage times 590
13.5 Barriers 595
13.6 Excursions and the Brownian bridge 599
13.7 Stochastic calculus 602
13.8 The Itô integral 605
13.9 Itô's formula 610
13.10 Option pricing 613
13.11 Passage probabilities and potentials 620
13.12 Problems 627

Appendix I. Foundations and notation 631

Appendix II. Further reading 636

Appendix III. History and varieties of probability 638

Appendix IV. John Arbuthnot's Preface to *Of the laws of chance* (1692) 641

Appendix V. Table of distributions 644

Appendix VI. Chronology 646

Bibliography 649

Notation 653

Index 655