PERIOD IS A CLASS PROPERTY

Theorem 0.1. *Intercommunicating states i, j have equal periods.*

Proof. For $k \in S$, let $D_k = \{m \geq 1 : p_{k,k}(m) > 0\}$, so that the period of k is $d_k = \gcd(D_k)$.

Let $i \leftrightarrow j$ and find $m, n \geq 1$ such that
$$\alpha := p_{i,j}(m)p_{j,i}(n) > 0.$$
By the Chapman–Kolmogorov equations,

(0.1) $$p_{i,i}(m + r + n) \geq \alpha p_{j,j}(r), \qquad r \geq 0.$$

By (0.1), $d_i \mid m + r + n$ for $r \in \{0\} \cup D_j$. In particular, $d_i \mid m + n$, and hence $d_i \mid r$ for $r \in D_j$. Therefore, $d_i \mid d_j$.

Similarly $d_j \mid d_i$, and therefore $d_i = d_j$. $\qquad\qquad\square$