Percolation in $\infty + 1$ Dimensions

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Abstract

We investigate percolation on the graph of the direct product $\mathbb{T} \times \mathbb{Z}$ of a regular tree $\mathbb{T}$ and the line $\mathbb{Z}$, in which each 'tree' edge is open with probability τ and each 'line' edge with probability λ. There are three non-trivial phases, corresponding to the existence of 0, ∞, and 1 infinite open clusters. Such results may be obtained also for the graph $\mathbb{T} \times \mathbb{Z}^d$ where $d \geq 2$.

1. Introduction

The mathematical theory of percolation was conceived by Simon Broadbent and John Hammersley three decades or so ago as a stochastic model for the flow of material through a porous medium (see Broadbent and Hammersley 1957). In more recent years it has been the subject of much attention from mathematicians and physicists, and progress has been great. Substantial advances have been made in the last ten years, during which time percolation theory has become established as a fundamental tool in modelling random media.

The two phases of percolation are now understood reasonably well. For bond percolation (say) on $\mathbb{Z}^d$, there is a critical density p_c of open edges with the property that if the actual density p satisfies $p < p_c$ then all open clusters are (a.s.) finite, whereas if $p > p_c$ then there exists (a.s.) a *unique* infinite open cluster. The majority of the main unanswered questions about percolation relate to the behaviour of the process when p is close or equal to p_c. The picture is somewhat different for the easier case of bond percolation on a regular tree. For such a graph (which is regarded as 'infinite-dimensional' by physicists), we learn from the theory of branching processes that there exists a critical density p_c ($= k^{-1}$, where $k + 1$ is the common degree of the vertices) such that for $p \leq p_c$ all open clusters are (a.s.) finite whereas if $p_c < p < 1$ then there exists (a.s.) *infinitely* many infinite open clusters. Thus for both lattice and tree there exist two phases; however, the corresponding supercritical phases differ qualitatively in the *number* of infinite clusters (*one* for the lattice, and *infinitely many* for the tree). One reason for this dichotomy lies in the fact that the growth function (i.e. the volume of the n-ball, or the number of vertices within distance n of the origin) grows polynomially (like n^d) for $\mathbb{Z}^d$ but exponentially for the tree (this is one of the reasons trees are sometimes thought of as infinite-dimensional). It is not difficult to see that, for a large class of graphs with periodic structures including all lattices and regular trees, the number N of infinite open clusters satisfies exactly one of $P_p(N=0)=1$, $P_p(N=1)=1$, $P_p(N=\infty)=1$, for any given value of p (see Newman and Schulman 1981). The existing proofs of the uniqueness of the infinite open cluster (Aizenman, Kesten, and Newman 1987; Gandolfi, Grimmett, and Russo 1988; Burton and Keane 1989) may be adapted to all 'periodic' graphs having the property that the surface-to-volume ratio of the n-ball tends to 0 as $n \to \infty$, and this covers all periodic graphs with sub-exponential growth functions; an interesting class of such graphs is discussed implicitly by Grigorchuk (1983).

Lattices and trees have two distinct phases. It is our purpose in this paper to explore the phase diagram of a graph which possesses (at least) three distinct phases, in which the number of infinite clusters is (a.s.) 0, ∞, and 1, respectively. The graph in question is the direct product of the line $\mathbb{Z}$ and a regular tree $\mathbb{T}$, and the actual construction is as follows. Let $\mathbb{T}$ be an infinite regular labelled tree with degree $k+1$, where $k \geq 2$. The *distance* $\delta_{\mathbb{T}}(t_1,t_2)$ between two vertices t_1 and t_2 is defined to be the number of edges in the unique path of $\mathbb{T}$ from t_1 to t_2. A nominated vertex of $\mathbb{T}$ is called the *origin* and labelled $\emptyset$ (the empty word). Vertices adjacent to $\emptyset$ are labelled $0,1,2,\ldots,k$ respectively. More generally, vertices having distance n (≥ 1) from the origin are labelled by words $\alpha_1\alpha_2\cdots\alpha_n$ where $\alpha_1 \in \{0,1,2,\ldots,k\}$ and $\alpha_i \in \{1,2,\ldots,k\}$ for $i \geq 2$; these labels are attached to the vertices in such a way that the vertex $\alpha_1\alpha_2\cdots\alpha_n$ has as neighbours the vertex $\alpha_1\alpha_2\cdots\alpha_{n-1}$ and $\alpha_1\alpha_2\cdots\alpha_n\alpha$ as α ranges over $\{1,2,\ldots,k\}$. We write $V(\mathbb{T})$ for the vertex set of $\mathbb{T}$. The second component of the graph under study is the line $\mathbb{Z}=\{z : z=\ldots,-1,0,1,\ldots\}$ with distance function $\delta_{\mathbb{Z}}(z_1,z_2)=|z_1-z_2|$. We denote by $\mathbb{L}$ the graph with vertex set $V(\mathbb{L})=\{(t,z) : t \in V(\mathbb{T}), z \in \mathbb{Z}\}$ and edge set given by the adjacency relation $(t_1,z_1) \sim (t_2,z_2)$ if and only if $\delta_{\mathbb{T}}(t_1,t_2)+\delta_{\mathbb{Z}}(z_1,z_2)=1$. We write $\mathbb{L}=\mathbb{T}\times\mathbb{Z}$ and note that two vertices of $\mathbb{L}$ are adjacent if and only if either their $\mathbb{T}$-components are equal and their $\mathbb{Z}$-components are adjacent in $\mathbb{Z}$, or vice versa. The *origin* of $\mathbb{L}$ is the vertex $(\emptyset,0)$. We call an edge of $\mathbb{L}$ a $\mathbb{T}$-edge (respectively a $\mathbb{Z}$-edge) if it joins two vertices which differ only in their $\mathbb{T}$-component (respectively $\mathbb{Z}$-component).

We shall consider bond percolation on $\mathbb{L}$, but rather than restricting ourselves to isotropic percolation with constant density, we allow a natural anisotropy as follows. Let τ and λ satisfy $0 \leq \tau,\ \lambda \leq 1$, and declare each $\mathbb{T}$-edge (respectively $\mathbb{Z}$-edge) to be open with probability τ (respectively λ) independently of the states of all other edges. We shall generally assume

FIG. 1. The set of possible values of (τ, λ) may be partitioned into three regions corresponding to the cases $N = 0$, $N = \infty$, and $N = 1$, respectively. The figure on the left is probably correct, although we have not ruled out the possibility that the figure on the right is correct for small values of k.

that $0 < \tau$, $\lambda < 1$ unless we state otherwise. We write $P_{\tau,\lambda}$ for the ensuing probability measure. Similarly we write P_τ and P_λ for the induced measures on the edge states of subgraphs of the form $\mathbb{T} \times \{z\}$ and $\{t\} \times \mathbb{Z}$ respectively for any given $z \in \mathbb{Z}$ and $t \in V(\mathbb{T})$. More generally, the individual subscripts τ and λ are used to denote quantities associated with projections of $\mathbb{L}$ onto copies of $\mathbb{T}$ or of $\mathbb{Z}$ respectively.

We shall explore the existence and number N of infinite open clusters in $\mathbb{L}$ for various ranges of values of the parameters (τ, λ). It is easy to show in the usual way that $P_{\tau,\lambda}(N = 0) = 1$ for all sufficiently small τ and λ. Also, it is not difficult to adapt the arguments of Newman and Schulman (1981) to see that, for any given (τ, λ), one of the following holds: (i) $N = 0$ a.s., (ii) $N = 1$ a.s., (iii) $N = \infty$ a.s. It turns out that the set of values of (τ, λ) (i.e. the unit square) may be partitioned into three regions each with non-empty interior corresponding to the three cases $N = 0$, $N = 1$, and $N = \infty$. See Figure 1.

We make a number of remarks about Figure 1:

1. The $\lambda = 0$ boundary of the unit square has of course $N = \infty$ for $1/k < \tau < 1$; among the results of this paper is that here $N = \infty$ is stable (respectively unstable) relative to $N = 1$ under perturbations of λ when $1/k < \tau < 1/\sqrt{k}$ (respectively $\tau \geq 1/\sqrt{k}$). Less interesting is the stability of $N = 0$ for $\lambda = 0$, $\tau < 1/k$ or for $\tau = 0$, $\lambda < 1$ and the stability of $N = 1$ for $\tau = 1$ or for $\lambda = 1$, $\tau > 0$.
2. The upper left point $(\tau, \lambda) = (0, 1)$, which has $N = \infty$ while being the endpoint of both $N = 0$ and $N = 1$ boundary segments, is clearly special. For values of k sufficiently large ($k \geq 6$ certainly suffices), the

FIG. 2. As in the case of Figure 1, the figure on the left is probably correct, although we have not ruled out the possibility of the figure on the right.

correct picture is the one on the left, in that any neighbourhood of the point $(\tau, \lambda) = (0, 1)$ contains points in each of the three regions. For small values of k, we have not ruled out the possibility that the second diagram in Figure 1 is correct (with only the $N = 0$ and $N = 1$ regions reaching to $(0, 1)$). The first diagram is of course correct for some smallest value of k, and we have no evidence that it is not always the correct picture.

3. A cautionary remark is that although Figure 1 shows the boundary between $N = \infty$ and $N = 1$ as the graph of a function, we are unaware of any (say monotonicity) argument which guarantees this. If the co-existence of a positive-density (defined using ergodicity in the $\mathbb{Z}$-direction) infinite cluster with infinitely many zero-density infinite clusters (as considered in Newman and Schulman 1981) could be ruled out, then monotonicity for the existence of a positive-density infinite cluster would yield such a conclusion. Note however that such a result would imply that along the common boundary of the $N = \infty$ and $N = 1$ regimes there is a line of discontinuities of $P_{\tau,\lambda}((\emptyset, 0)$ belongs to a positive-density infinite cluster)!

A somewhat different picture emerges if the $\mathbb{Z}$-component of $\mathbb{L}$ is replaced by the d-dimensional cubic lattice $\mathbb{Z}^d$ where $d \geq 2$, since such a lattice is capable of sustaining an infinite open cluster without support from the $\mathbb{T}$-edges. Our analysis may be adapted to this situation at little extra cost, and we believe that the correct phase diagram, at least for sufficiently large values of k, is as drawn on the left side of Figure 2, although for no value of k have we ruled out the possibility that the right-hand picture is correct. In this figure, $\lambda_c(d)$ is the critical value of λ for bond percolation on $\mathbb{Z}^d$.

This paper is laid out in the following way. In Section 2 we introduce the necessary notation and we review some useful facts about percolation theory. In Section 3 we explore conditions which are (respectively) necessary and sufficient for the (a.s.) existence of an infinite open cluster in $\mathbb{L}$. This amounts to finding lower and upper bounds for the lower curve in Figure 1. We turn then to conditions which are (respectively) necessary and sufficient for the (a.s.) existence of infinitely many infinite open clusters; this amounts to establishing upper and lower bounds for the upper curve in Figure 1. We present such results in Section 4. There follows a final section devoted to the graph $\mathbb{T} \times \mathbb{Z}^d$ where $d \geq 2$. We remark that results similar to those of this paper may be derived for Ising (and Potts) models on $\mathbb{T} \times \mathbb{Z}$ and $\mathbb{T} \times \mathbb{Z}^d$ (Wu 1989; Newman and Wu 1989).

2. Percolation Notation and Background

For any graph G, we write $V(G)$ for its vertex set and $\langle u, v \rangle$ for the edge between neighbours u and v. We shall explore percolation on $\mathbb{L}$ and on the square lattice $\mathbb{Z}^2$, and have already defined the appropriate percolation model on $\mathbb{L}$. For $\mathbb{Z}^2$ we shall be interested in anisotropic percolation in which each edge $\langle (z_1, z_2), (z_1 + 1, z_2) \rangle$ is open with probability τ, and each edge $\langle (z_1, z_2), (z_1, z_2 + 1) \rangle$ is open with probability λ. We write P and E for the corresponding probability measure and expectation.

In studying percolation on $\mathbb{L}$, we shall use certain results about percolation on $\mathbb{Z}^2$. It is easy to see why such results are relevant. Let π be a doubly infinite path in $\mathbb{T}$ (paths are defined to be self-avoiding). Then π induces a subgraph of $\mathbb{L}$, viz. that with vertex set $\Pi_\pi = \{(t, z) : t \in V(\pi), z \in \mathbb{Z}\}$, and it is easily seen that this subgraph is isomorphic to $\mathbb{Z}^2$; furthermore each 'horizontal' edge of Π_π is open with probability τ, and each 'vertical' edge with probability λ.

We write $\theta(\tau, \lambda)$ for the probability that the origin of $\mathbb{Z}^2$ is in an infinite open cluster of $\mathbb{Z}^2$ and $\theta_{\mathbb{L}}(\tau, \lambda)$ for the probability that the origin of $\mathbb{L}$ is in an infinite open cluster of $\mathbb{L}$.

It is well known that $\theta(\tau, \lambda) > 0$ if and only if $\tau + \lambda > 1$, and that the infinite open cluster is (a.s.) unique under this assumption. See Kesten (1982) and Grimmett (1989) for these and related results and techniques. It is completely standard that there exists (a.s.) an infinite open cluster in any connected graph G if and only if each vertex of G has a strictly positive probability of being in such a cluster. For subsets A and B of the vertex set of G, we write $A \leftrightarrow B$ if there exists an open path in G joining some vertex in A to some vertex in B.

We shall need the idea of the (horizontal) correlation length of bond percolation on $\mathbb{Z}^2$. Define the 'strip'

$$T_m(n) = \{(z_1, z_2) \in \mathbb{Z}^2 : 0 \leq z_1 \leq n, |z_2| \leq m\} \tag{2.1}$$

of length n and height $2m$, and turn $T_m(n)$ into a graph by adding all appropriate edges of $\mathbb{Z}^2$ except those in the 'left' and 'right' sides of $T_m(n)$ (i.e. those of the form $\langle (0,y),(0,y+1)\rangle$ and $\langle (n,y),(n,y+1)\rangle$). Let

$$\phi_m(\tau,\lambda) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P\big((0,0) \leftrightarrow (n,0) \text{ in } T_m(n)\big) \right\}. \tag{2.2}$$

The limit exists by subadditivity, and furthermore

$$P\big((0,0) \leftrightarrow (n,0) \ \text{ in } \ T_m(n)\big) \le e^{-n\phi_m(\tau,\lambda)} \ \text{ for all } \ n. \tag{2.3}$$

As in the case of isotropic percolation on $\mathbb{Z}^2$ (see Aizenman, Chayes, Chayes, and Newman 1988 and Grimmett 1989) it is the case that

$$\phi_m(\tau,\lambda) \downarrow \phi(\tau,\lambda) \ \text{ as } \ m \to \infty \tag{2.4}$$

where

$$\phi(\tau,\lambda) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P\big((0,0) \leftrightarrow (n,0)\big) \right\} \tag{2.5}$$

is the reciprocal of the (horizontal) correlation length. Note that, as usual, $\phi(\tau,\lambda)$ is strictly decreasing in τ and λ when $\tau+\lambda<1$, and

$$\phi(\tau,\lambda) \downarrow 0 \ \text{ as } \ \lambda \uparrow 1-\tau \ \text{ or } \ \tau \uparrow 1-\lambda; \tag{2.6}$$

see Grimmett (1989, Ch. 5) for the corresponding results for isotropic percolation. We note that ϕ may be defined equivalently by

$$\phi(\tau,\lambda) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P\big((0,0) \leftrightarrow L_n \text{ in } H\big) \right\} \tag{2.7}$$

where L_n is the vertical line $\{(n,z) : z \in \mathbb{Z}\}$ and H is the half-plane $\{(x,y) : x \ge 0, y \in \mathbb{Z}\}$.

Inequality (2.3) provides an upper bound for

$$p_m(n) = P\big((0,0) \leftrightarrow (n,0) \text{ in } T_m(n)\big). \tag{2.8}$$

A comparable lower bound is easily obtained as follows. Let r and s be positive integers, and let A be the event that all edges of the form $\langle (r-1,y),(r-1,y+1)\rangle, \langle (r+1,y),(r+1,y+1)\rangle$ for $-m \le y < m$ together with the edges $\langle (r-1,0),(r,0)\rangle$ and $\langle (r,0),(r+1,0)\rangle$ are open. We have by the FKG inequality that

$$\lambda^{4m}\tau^2 p_m(r+s) \le P\big((0,0) \leftrightarrow (r+s,0) \ \text{ in } \ T_m(r+s), \text{ and } A\big).$$

The latter probability is no greater than $p_m(r)p_m(s)$, so that

$$\lambda^{4m}\tau^2 p_m(r+s) \le p_m(r)p_m(s),$$

implying by standard arguments that

$$p_m(n) \ge \lambda^{4m}\tau^2 e^{-n\phi_m(\tau,\lambda)} \quad \text{for all} \quad n. \tag{2.9}$$

Finally, here are two bounds involving the function $\phi(\tau,\lambda)$. Clearly $p_m(n) \ge \tau^n$, so that

$$e^{-\phi(\tau,\lambda)} \ge \tau. \tag{2.10}$$

It is not hard to improve this in the following standard way. It is sometimes possible to find open connections (in H) from the origin to the line L_n by observing the (possibly empty) vertical line of open edges through the origin, finding some open edge leading rightwards from this line, and so on. The probability that this construction succeeds in reaching the line L_n is $\{1 - E((1-\tau)^L)\}^n$ where L is the number of vertices in the vertical line of open edges through the origin. An easy calculation shows that

$$E\big((1-\tau)^L\big) = (1-\tau)\left\{\frac{1-\lambda}{1-\lambda(1-\tau)}\right\}^2$$

and hence

$$e^{-\phi(\tau,\lambda)} \ge 1 - \frac{(1-\tau)(1-\lambda)^2}{(1-\lambda(1-\tau))^2}. \tag{2.11}$$

3. Existence of Infinite Clusters in $\mathbb{L}$

Our first results provide (respectively) necessary and sufficient conditions for the existence in $\mathbb{L}$ of an infinite open cluster.

Proposition 1. *If*

$$\tau k(1 + \lambda + \sqrt{2\lambda(1+\lambda)}) < 1 - \lambda \tag{3.1}$$

then there is a.s. no infinite open cluster in $\mathbb{L}$.

Proposition 2. *If*

$$ke^{-\phi(\tau,\lambda)} > 1 \tag{3.2}$$

where ϕ *is given by (2.5) or (2.7), then there is a.s. an infinite open cluster in* $\mathbb{L}$.

Before giving their proofs, we make a number of remarks concerning these two propositions:

1. Proposition 1 provides a lower bound $\tau = \underline{\tau}_l(\lambda)$ for the lower curve of Figure 1 with the endpoint properties
$$\underline{\tau}_l(0) = \frac{1}{k}, \quad \underline{\tau}_l'(0) = -\infty, \quad \underline{\tau}_l(1) = 0, \quad \text{and} \quad \underline{\tau}_l'(1) = -\frac{1}{4k}. \tag{3.3}$$
2. Proposition 2 provides an upper bound $\tau = \overline{\tau}_l(\lambda)$ for the same curve with a quality depending on how accurate a bound is used for $\phi(\tau, \lambda)$. The trivial bound (2.10) yields the obvious result that $\tau > 1/k$ implies percolation. The improved bound (2.11) for $\phi(\tau, \lambda)$ gives a $\overline{\tau}_l(\lambda)$ with
$$\begin{gathered} \overline{\tau}_l(0) = \frac{1}{k}, \quad \overline{\tau}_l'(0) = -2(k-1)/k^2, \\ \overline{\tau}_l(1) = 0, \quad \text{and} \quad \overline{\tau}_l'(1) = -\left(\sqrt{\frac{k}{k-1}} - 1\right). \end{gathered} \tag{3.4}$$
3. Various small improvements in Proposition 1, (2.11), and Proposition 2 may be obtained. We do not present these here because the increased cost in their proofs seems to be out of proportion to the information gained.

Proof of Proposition 1: We shall find an upper bound for $\mathcal{X}(\tau, \lambda)$, the mean number of vertices in the open cluster C of $\mathbb{L}$ at the origin. Our target is to show that $\mathcal{X}(\tau, \lambda) < \infty$ if (3.1) holds, since this implies that C is a.s. finite. We shall make use later of the same method of proof when finding a condition which guarantees the a.s. existence of infinitely many infinite clusters. Clearly
$$\mathcal{X}(\tau, \lambda) = \sum_{(t,z) \in V(\mathbb{L})} P_{\tau,\lambda}((\emptyset, 0) \leftrightarrow (t, z)). \tag{3.5}$$
Now $(\emptyset, 0) \leftrightarrow (t, z)$ if and only if there exists a (self-avoiding) path of $\mathbb{L}$ from $(\emptyset, 0)$ to (t, z) which is open. Any such path contains $\mathbb{T}$-edges and $\mathbb{Z}$-edges, but may be projected onto the section $\mathbb{T} \times \{0\}$ to give a route from $(\emptyset, 0)$ to the vertex $(t, 0)$ (*routes* are paths with the self-avoiding condition removed). This route is a sequence $(t_0, 0), (t_1, 0), \ldots, (t_n, 0)$ where $t_0 = \emptyset$, $t_n = t$, and t_i is adjacent to t_{i+1} in $\mathbb{T}$ for $0 \le i < n$. We denote this route by $\pi_{\mathbf{t}} = (t_0, t_1, \ldots, t_n)$, which we think of as a route in $\mathbb{T}$ from $\emptyset$ to t. The aforesaid path in $\mathbb{L}$ from $(\emptyset, 0)$ to (t, z) proceeds along $\mathbb{Z}$-edges from $(\emptyset, 0)$ to some $(\emptyset, z_0)$, thence along $\mathbb{T}$-edges to (t_1, z_0), thence along $\mathbb{Z}$-edges to some (t_1, z_1), thence to (t_2, z_1), and so on until it arrives at $(t_n, z_n) = (t, z)$. We denote this path by $\pi(\mathbf{t}, \mathbf{z})$. It follows from (3.5) that
$$\begin{aligned} \mathcal{X}(\tau, \lambda) &\le \sum_{(t,z)} \sum_{\substack{\mathbf{t}: \\ t_n = t}} \sum_{\substack{\mathbf{z}: \\ z_n = z}} P_{\tau,\lambda}(\pi(\mathbf{t}, \mathbf{z}) \text{ is open}) \\ &= \sum_{n=0}^{\infty} \sum_{\mathbf{t}} \sum_{\mathbf{z}} P_{\tau,\lambda}(\pi(\mathbf{t}, \mathbf{z}) \text{ is open}) \end{aligned} \tag{3.6}$$

where the final two summations are over all appropriate sequences $\mathbf{t} = (t_0, t_1, \ldots, t_n)$, $\mathbf{z} = (z_0, z_1, \ldots, z_n)$, where $(t_0, z_0) = (\emptyset, 0)$. We sum first over possible values for z_n. The number of choices for z_n is restricted by the fact that $\pi(\mathbf{t}, \mathbf{z})$ may already have visited the line $\{t_n\} \times \mathbb{Z}$ thereby removing certain possible vertices from consideration. In any case, the set of possibilities for z_n is no larger than the whole line $\{t_n\} \times \mathbb{Z}$, so that

$$\sum_{z_n} P_{\tau,\lambda}(\pi(\mathbf{t}, \mathbf{z}) \text{ is open}) \leq \mathcal{X}_\lambda P_{\tau,\lambda}(\pi(\mathbf{t}, \mathbf{z})' \text{ is open}) \tag{3.7}$$

where

$$\mathcal{X}_\lambda = 1 + 2\sum_{i=1}^{\infty} \lambda^i = \frac{1+\lambda}{1-\lambda} \tag{3.8}$$

is the mean number of vertices on $\{t_n\} \times \mathbb{Z}$ joined by open $\mathbb{Z}$-paths to (t_n, z_{n-1}), and $\pi(\mathbf{t}, \mathbf{z})'$ is the path $\pi(\mathbf{t}, \mathbf{z})$ with the $\mathbb{Z}$-edges from (t_n, z_{n-1}) to (t_n, z_n) removed. Progressive summation over all the z_i's yields similarly

$$\mathcal{X}(\tau, \lambda) \leq \sum_{n=0}^{\infty} \sum_{\mathbf{t}} \tau^n \mathcal{X}_\lambda^{n+1}, \tag{3.9}$$

the term τ^n coming from the fact that $\pi(\mathbf{t}, \mathbf{z})$ uses exactly n $\mathbb{T}$-edges of $\mathbb{L}$. This bound for $\mathcal{X}(\tau, \lambda)$ may be improved as follows. For a given route t_0, $t_1, \ldots, t_n$ with $t_0 = \emptyset$, define for $2 \leq i \leq n$

$$I_i = \begin{cases} 1 & \text{if } t_i = t_{i-2}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S(\mathbf{t}) = \sum_{i=2}^{n} I_i.$$

If $I_n = 1$ then the projected walk $\pi_{\mathbf{t}}$ moves from t_{n-2} to t_{n-1} and back to t_{n-2} $(= t_n)$. In this circumstance, the sum over possible choices for z_{n-1} in (3.6) contributes no more than $\mathcal{X}_\lambda - 1$, since the path $\pi(\mathbf{t}, \mathbf{z})$ is self-avoiding and thus $z_{n-1} \neq z_{n-2}$. It follows similarly that

$$\begin{aligned} \mathcal{X}(\tau, \lambda) &\leq \sum_{n=0}^{\infty} \sum_{\mathbf{t}} \tau^n (\mathcal{X}_\lambda - 1)^{S(\mathbf{t})} \mathcal{X}_\lambda^{n+1-S(\mathbf{t})} \\ &= \mathcal{X}_\lambda \sum_{n=0}^{\infty} (\tau \mathcal{X}_\lambda)^n \sum_{\mathbf{t}} (1 - \mathcal{X}_\lambda^{-1})^{S(\mathbf{t})} \end{aligned} \tag{3.10}$$

in place of (3.9).

Let π be a route in $\mathbb{T}$ beginning at the origin, thought of as a sequence of *directed* steps. We classify each step of π as either an 'outstep' or an 'instep' according to the following rule. A step from t_1 to t_2 where $t_1 \neq \emptyset$ is an *outstep* if and only if $\delta_{\mathbb{T}}(\emptyset, t_2) > \delta_{\mathbb{T}}(\emptyset, t_1)$. A step from $\emptyset$ to t is an *outstep* if t is not labelled 0 and an *instep* otherwise. From each $t \in \mathbb{T}$ there are exactly k possible outsteps.

Returning to the route $\pi_{\mathbf{t}}$ above, we define $J_i = 1$ if the step of $\pi_{\mathbf{t}}$ from t_i to t_{i+1} is an instep and $J_i = 2$ otherwise. We set

$$T(\mathbf{J}) = |\{i : J_{i-1} = 2, J_i = 1\}|,$$

the number of times an outstep is followed by an instep. Note that $T(\mathbf{J}) \leq S(\mathbf{t})$ so that

$$\begin{aligned} \mathcal{X}(\tau, \lambda) &\leq \mathcal{X}_\lambda \sum_{n=0}^{\infty} (\tau \mathcal{X}_\lambda)^n \sum_{\mathbf{t}} (1 - \mathcal{X}_\lambda^{-1})^{T(\mathbf{J})} \\ &\leq \mathcal{X}_\lambda \sum_{n=0}^{\infty} (k\tau \mathcal{X}_\lambda)^n \sum_{\mathbf{J}} (1 - \mathcal{X}_\lambda^{-1})^{T(\mathbf{J})} \end{aligned} \tag{3.11}$$

where the final summation is over all sequences $\mathbf{J} = (J_0, J_1, \cdots, J_{n-1})$ of 1's and 2's; the second inequality holds since each sequence $\mathbf{J}$ corresponds to at most k^n sequences $\mathbf{t}$. However,

$$\sum_{\mathbf{J}} (1 - \mathcal{X}_\lambda^{-1})^{T(\mathbf{J})} = (\,1 \quad 1\,) \begin{pmatrix} 1 & 1 \\ 1 - \mathcal{X}_\lambda^{-1} & 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which behaves for large n in the manner of η^{n-1} where $\eta = 1 + \sqrt{1 - \mathcal{X}_\lambda^{-1}}$ is the larger eigenvalue of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 - \mathcal{X}_\lambda^{-1} & 1 \end{pmatrix}.$$

Hence $\mathcal{X}(\tau, \lambda) < \infty$ if

$$k\tau \mathcal{X}_\lambda \big(1 + \sqrt{1 - \mathcal{X}_\lambda^{-1}}\big) < 1. \tag{3.12}$$

Substituting from (3.8) for $\mathcal{X}_\lambda$, we obtain the assertion of the proposition. ■

Proof of Proposition 2: We shall show that $\theta_{\mathbb{L}}(\tau, \lambda) > 0$ if $ke^{-\phi(\tau,\lambda)} > 1$. Consider the subtree $\mathbb{T}^+$ of $\mathbb{T}$ being the component containing $\emptyset$ of the graph obtained from $\mathbb{T}$ by deleting the edge $\langle \emptyset, 0 \rangle$, and fix a positive integer

L; later we shall take the limit as $L \to \infty$. We construct a branching process on $\mathbb{T}^+$ as follows. If $t \in V(\mathbb{T}^+)$ is such that $\delta_{\mathbb{T}}(\emptyset, t) = L$, we declare t to be *green* if there exists an open path of $\mathbb{T}^+ \times \mathbb{Z}$ joining $(\emptyset, 0)$ to $(t, 0)$ and (apart from its endvertices) using only vertices of $\mathbb{L}$ in $\{(u,z) \in V(\mathbb{L}) : 0 < \delta_{\mathbb{T}}(\emptyset, u) < L, \quad z \in \mathbb{Z}\}$. Proceeding inductively, suppose that $t \in V(\mathbb{T}^+)$ is such that $\delta_{\mathbb{T}}(\emptyset, t) = aL$ for some positive integer a. There is a unique $v \in V(\mathbb{T}^+)$ such that $\delta_{\mathbb{T}}(\emptyset, v) = (a-1)L$, $\delta_{\mathbb{T}}(v, t) = L$. We declare t to be *green* if and only if (i) v is green, and (ii) there is an open path from $(v, 0)$ to $(t, 0)$ using (apart from its endvertices) only vertices of $\mathbb{L}$ in $\{(u, z) : (a-1)L < \delta_{\mathbb{T}}(\emptyset, u) < a\mathbb{L}, \quad z \in \mathbb{Z}\}$. It should be clear that the set of green vertices constitutes a branching process on $\mathbb{T}^+$ with mean family-size at least

$$\mu_L = k^L P\big((0,0) \leftrightarrow (L,0) \text{ in } T_\infty(L)\big)$$

where P is the probability measure of anisotropic percolation on $\mathbb{Z}^2$ and

$$T_\infty(L) = \lim_{m \to \infty} T_m(L),$$

$T_m(L)$ being the strip given in (2.1). If $\mu_L > 1$ for some $L \geq 1$ then this branching process is supercritical and has therefore strictly positive probability of being infinite. This implies that $\theta_{\mathbb{L}}(\tau, \lambda) > 0$. Now

$$\begin{aligned} -\frac{1}{L}\log \mu_L &= -\log k - \frac{1}{L}\log P\big((0,0) \leftrightarrow (L,0) \text{ in } T_\infty(L)\big) \\ &\leq -\log k - \frac{1}{L}\log P\big((0,0) \leftrightarrow (L,0) \text{ in } T_m(L)\big) \\ &\to -\log k + \phi_m(\tau, \lambda) \qquad \text{as } \ L \to \infty \\ &\to -\log(k e^{-\phi(\tau,\lambda)}) \qquad \text{as } \ m \to \infty \end{aligned}$$

by (2.2) and (2.4). Therefore, if $ke^{-\phi(\tau,\lambda)} > 1$ then $\mu_L > 1$ for all large L, and the result is proved. ∎

4. Existence of Infinitely Many Infinite Clusters

We establish in Propositions 4 and 5 below conditions which are (respectively) sufficient and necessary for the existence of infinitely many infinite open clusters in $\mathbb{L}$. First we state a lemma relating this phenomenon to the decay of the connectivity function; its proof is given after the statement of Proposition 5.

LEMMA 3. *Let C be the open cluster of $\mathbb{L}$ at the origin, and for $t \in V(\mathbb{T})$ let*

$$D_t = \{(t, z) \in \mathbb{L} : (\emptyset, 0) \leftrightarrow (t, z)\}$$

denote the intersection of C with $\{t\} \times \mathbb{Z}$. If, for some τ, λ, it is the case that $|D_\emptyset| < \infty$ a.s., then
(i) $|D_t| < \infty$ a.s., for all $t \in V(\mathbb{T})$,
(ii) $P_{\tau,\lambda}((\emptyset, 0) \leftrightarrow (t, z)) \to 0$ as $\delta_{\mathbb{T}}(\emptyset, t) + |z| \to \infty$.
If further $\theta_{\mathbb{L}}(\tau, \lambda) > 0$, then there are a.s. infinitely many infinite open clusters in $\mathbb{L}$, each of which intersects each $\{t\} \times \mathbb{Z}$ in only finitely many vertices.

PROPOSITION 4. *If*

$$\tau\sqrt{k}\big(1 + \lambda + \sqrt{2\lambda(1+\lambda)}\big) < 1 - \lambda, \tag{4.1}$$

then $|D_\emptyset| < \infty$ a.s. Thus if in addition $\theta_{\mathbb{L}}(\tau, \lambda) > 0$, then there exist a.s. infinitely many infinite open clusters in $\mathbb{L}$.

Note that (4.1) differs from (3.1) only in the replacement of k by $\sqrt{k}$. As was the case with (3.1), we may improve condition (4.1) to obtain a weaker condition sufficient for the conclusion. Such improvements incur extra costs without the benefit of substantial improvement towards optimality. Proposition 4 provides a lower bound $\tau = \underline{\tau}_u(\lambda)$ for the upper curve of Figure 1 satisfying

$$\begin{aligned} \underline{\tau}_u(0) &= \frac{1}{\sqrt{k}}, \quad \underline{\tau}_u'(0) = -\infty, \\ \underline{\tau}_u(1) &= 0, \quad \text{and} \quad \underline{\tau}_u'(1) = -\frac{1}{4\sqrt{k}}. \end{aligned} \tag{4.2}$$

Proposition 4 combined with Proposition 2 implies the existence for all $k \geq 2$ of a region of values of (τ, λ) for which there exist infinitely many infinite open clusters; to see this, simply note from (3.4) and (4.2) that $\overline{\tau}_l(0) < \underline{\tau}_u(0)$. For sufficiently large k (i.e. $k \geq 6$) our estimates imply that this region extends all the way to the point $(\tau, \lambda) = (0, 1)$ since $|\overline{\tau}_l'(1)| < |\underline{\tau}_u'(1)|$ for $k \geq 6$. It can also be checked that, for large k, $\overline{\tau}_l(\lambda) < \underline{\tau}_u(\lambda)$ for all $0 < \lambda < 1$.

We turn next to conditions which are sufficient for the a.s. uniqueness of the infinite cluster. It is not too difficult to show that there is a.s. a unique infinite open cluster when $\tau + \lambda > 1$, making use of the fact that each infinite path π in $\mathbb{T}$ gives rise to a subgraph $\Pi_\pi = \{(t, z) : t \in V(\pi), z \in \mathbb{Z}\}$ of $\mathbb{L}$ which is isomorphic to $\mathbb{Z}^2$ and therefore contains a.s. a unique infinite open cluster. We weaken the condition $\tau + \lambda > 1$ in the next proposition.

PROPOSITION 5. *If*

$$ke^{-2\phi(\tau,\lambda)} > 1 \tag{4.3}$$

then there exists a.s. a unique infinite open cluster in $\mathbb{L}$.

As with Proposition 4, the condition (4.3) of Proposition 5 differs from (3.2), the corresponding condition for actual percolation, in the replacement by $\sqrt{k}$ of k. Proposition 5 combined with (2.11) provides an upper bound $\overline{\tau}_u(\lambda)$ for the upper curve of Figure 1 satisfying

$$\begin{gathered}\overline{\tau}_u(0) = \frac{1}{\sqrt{k}}, \quad \overline{\tau}'_u(0) = -2(\sqrt{k}-1)/k, \\ \overline{\tau}_u(1) = 0, \quad \text{and} \quad \overline{\tau}'_u(1) = -\left(\left(\frac{\sqrt{k}}{\sqrt{k}-1}\right)^{1/2} - 1\right).\end{gathered} \tag{4.4}$$

We remark that it is natural to conjecture that when $\theta_{\mathbb{L}}(\tau, \lambda) > 0$, either the infinite cluster is unique or else the situation of Lemma 3 is valid; however this has not been proved.

Proof of Lemma 3: We first prove (i). Let us suppose that

$$P_{\tau,\lambda}(|D_\emptyset| < \infty) = 1$$

but that there exist $t \in \mathbb{T}$ such that

$$P_{\tau,\lambda}(|D_t| = \infty) > 0.$$

We may choose such a t such that $\delta_{\mathbb{T}}(\emptyset, t) = m$ is a minimum, and we write s for the unique vertex of $\mathbb{T}$ satisfying $\delta_{\mathbb{T}}(\emptyset, s) = m-1$, $\delta_{\mathbb{T}}(s,t) = 1$. Then D_s is a.s. finite but $P_{\tau,\lambda}(|D_t| = \infty) = \eta > 0$; we shall show the event $\{|D_s| < \infty\} \cap \{|D_t| = \infty\}$ has probability zero, thus contradicting the minimality of $\delta_{\mathbb{T}}(\emptyset, t)$. Pick ϵ satisfying $0 < \epsilon < \eta$ and find a positive integer M such that

$$P_{\tau,\lambda}\big((\emptyset, 0) \leftrightarrow (s, z) \text{ for some } |z| > M\big) < \epsilon. \tag{4.5}$$

On the other hand

$$P_{\tau,\lambda}\big((\emptyset, 0) \leftrightarrow (t, z) \text{ for infinitely many } |z| > M\big) = \eta.$$

There is probability at least $\eta - \epsilon$ that there exists an infinite set Z of integers z with $|z| > M$ such that $(\emptyset, 0) \leftrightarrow (t, z)$ for all $z \in Z$ in the graph obtained from $\mathbb{L}$ by deleting (i.e. without examining the states of) the edges in the set $E = \{\langle (s,z), (t,z) \rangle : |z| > M\}$. Almost surely infinitely many edges in E having an endvertex of the form (t, z) for $z \in Z$ are open. Hence, $|D_s| = \infty$ occurs with probability $\eta - \epsilon > 0$, a contradiction.

Before discussing (ii) we prove the final statement of the lemma. If there were a.s. a unique infinite open cluster, then by ergodicity in the $\mathbb{Z}$-direction the set of $(\emptyset, z)$ belonging to this cluster would have positive density $\theta_{\mathbb{L}}$ and hence would be infinite, so that $D_\emptyset$ would be infinite with probability $\theta_{\mathbb{L}} > 0$, a contradiction. However, by the arguments of Newman and Schulman (1981), the only alternative is that there exist a.s. infinitely many infinite clusters; each such cluster must have a.s. finite intersection with $\{t\} \times \mathbb{Z}$ by (i).

It remains to prove (ii). We first note that by (i) the probability in (ii) tends to zero as $|z| \to \infty$ for fixed t. Thus we assume that for some sequence (t_i, z_i) with $\delta_{\mathbb{T}}(\emptyset, t_i) \to \infty$, it is the case that

$$P_{\tau,\lambda}\big((\emptyset, 0) \leftrightarrow (t_i, z_i)\big) \geq \eta > 0 \quad \text{for all} \quad i,$$

and we search for a contradiction. For each i, we may choose R_i so that for any z, the event A_i^z, that $(\emptyset, z) \leftrightarrow (t_i, z_i + z)$ in the region $\mathcal{R}_i = \{(t, z') : \delta_{\mathbb{T}}(\emptyset, t) \leq R_i, z' \in \mathbb{Z}\}$, is such that $P_{\tau,\lambda}(A_i^z) \geq \eta/2$. By choosing a subsequence if necessary, we can and will assume that $R_i < \delta_{\mathbb{T}}(\emptyset, t_{i+1})$ for each i. For $z \geq 0$, let B_i^z be the event that both $(\emptyset, 0)$ and $(\emptyset, z)$ are connected in the region $\mathcal{R}_i$ to vertices in $\{(t_i, z') : z_i \leq z' \leq z_i + z\}$. Then by the Harris-FKG inequality,

$$P_{\tau,\lambda}(B_i^z) \geq P_{\tau,\lambda}(A_i^0 \cap A_i^z) \geq (\eta/2)^2 \quad \text{for all} \quad z \geq 0$$

so that B_i^z occurs for infinitely many i's with probability at least $(\eta/2)^2$. We show next that

$$\begin{aligned} P_{\tau,\lambda}\big((\emptyset, 0) \leftrightarrow (\emptyset, z)\big) &\geq P_{\tau,\lambda}(B_i^z \text{ occurs for infinitely many } i\text{'s}) \\ &\geq (\eta/2)^2; \end{aligned} \tag{4.6}$$

the contradiction follows since we have already concluded from (i) that $P_{\tau,\lambda}((\emptyset, 0) \leftrightarrow (\emptyset, z)) \to 0$ as $|z| \to \infty$.

To obtain (4.6) let us for a given z define $b_1, b_2, \ldots$ to be the sequence of i's for which B_i^z occurs ($b_k = \infty$ if B_i^z occurs fewer then k times). Then (4.6) is an easy consequence of the limit as $j \to \infty$ of the inequalities

$$\begin{aligned} &P_{\tau,\lambda}\big(b_j < \infty \text{ and } (\emptyset, 0) \not\leftrightarrow (\emptyset, z) \text{ in } \mathcal{R}_{b_j}\big) \\ &\qquad \leq (1 - \lambda^z) P_{\tau,\lambda}\big(b_{j-1} < \infty \text{ and } (\emptyset, 0) \not\leftrightarrow (\emptyset, z) \text{ in } \mathcal{R}_{b_{j-1}}\big) \\ &\qquad \leq (1 - \lambda^z)^j. \end{aligned}$$

To obtain this estimate, condition on b_j and on the states of all edges in $\mathcal{R}_{b_j}$ except the z $\mathbb{Z}$-edges connecting the vertices in $\{(t_{b_j}, z') : z_i \leq z' \leq z_i + z\}$. The states of these z edges are independent of the value of b_j, and

$(\emptyset, 0) \leftrightarrow (\emptyset, z)$ in $\mathcal{R}_{b_j}$ if all z edges are open, which occurs with probability λ^z. The desired estimate follows. ∎

PROOF OF PROPOSITION 4: The proof resembles very closely that of Proposition 1. Arguing as in that proof, we find that

$$E_{\tau,\lambda}|D_\emptyset| \le \mathcal{X}_\lambda \sum_{n=0}^{\infty} (\tau \mathcal{X}_\lambda)^n \sum_{\mathbf{t}} (1 - \mathcal{X}_\lambda^{-1})^{T(\mathbf{J})}$$

where, unlike (3.11), the second summation is over all routes $\mathbf{t} = (t_0, t_1, \dots, t_n)$ in $\mathbb{T}$ satisfying $t_0 = \emptyset$ and $t_n = \emptyset$. In such a case, n is even. We claim that any such path contains no more than $\frac{1}{2}n$ outsteps. To see this, note that in excursions of $\mathbf{t}$ from $\emptyset$ beginning with an outstep, the number of outsteps equals the number of insteps, whereas in excursions beginning with an instep, the insteps outnumber the outsteps by 2. Hence each sequence $\mathbf{J}$ corresponds to at most $k^{n/2}$ sequences $\mathbf{t}$, giving that

$$E_{\tau,\lambda}|D_\emptyset| \le \mathcal{X}_\lambda \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} (\tau \mathcal{X}_\lambda \sqrt{k})^n \sum_{\mathbf{J}} (1 - \mathcal{X}_\lambda^{-1})^{T(\mathbf{J})}$$

which, by the previous argument, converges if

$$\tau \mathcal{X}_\lambda \sqrt{k}\big(1 + \sqrt{1 - \mathcal{X}_\lambda^{-1}}\big) < 1.$$

Substituting $\mathcal{X}_\lambda = (1+\lambda)/(1-\lambda)$, we conclude that, under (4.1), $E_{\tau,\lambda}|D_\emptyset| < \infty$ and therefore $P_{\tau,\lambda}(|D_\emptyset| = \infty) = 0$ as required. ∎

PROOF OF PROPOSITION 5: We suppose that

$$ke^{-2\phi} > 1. \tag{4.7}$$

By (2.4), we may pick a positive integer m such that

$$ke^{-2\phi_m} > 1$$

where

$$\phi_m(\tau, \lambda) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P\big((0,0) \leftrightarrow (n,0) \text{ in } T_m(n)\big) \right\}$$

as in (2.2). We have from (2.9) that

$$p_m(n) = P\big((0,0) \leftrightarrow (n,0) \ \text{ in } \ T_m(n)\big)$$

satisfies

$$p_m(n) \geq \lambda^{4m}\tau^2 e^{-n\phi_m} \quad \text{for all} \quad n \geq 0,$$

and therefore we may pick a positive integer n such that

$$k^n p_m(n)^2 \geq \lambda^{8m}\tau^4(ke^{-2\phi_m})^n > 1. \tag{4.8}$$

Next we recall and introduce some notation. For any vertex (t,z) of $\mathbb{L}$ with $t \neq \emptyset$, there is a unique vertex s of $\mathbb{T}$ such that $\delta_{\mathbb{T}}(s,\emptyset) = \delta_{\mathbb{T}}(t,\emptyset) - 1$ and $\delta_{\mathbb{T}}(s,t) = 1$. Let $\mathbb{T}^+(t)$ denote the subtree of $\mathbb{T}$ being the component containing t of the graph obtained by deleting from $\mathbb{T}$ the edge $\langle s,t\rangle$. We write $\mathbb{T}^+(t,z)$ for the subgraph of $\mathbb{L}$ induced by the vertex set $\{(u,z) : u \in V(\mathbb{T}^+(t))\}$, and we denote by $\mathbb{T}^+_m(t,z)$ the subgraph of $\mathbb{L}$ induced by the vertex set $\{(u,y) : u \in V(\mathbb{T}^+(t)), |z-y| \leq m\}$. We introduce similar notation for a vertex of the form $(\emptyset, z)$ in terms of the tree $\mathbb{T}^+(\emptyset)$ obtained as the component containing $\emptyset$ of the graph obtained from $\mathbb{T}$ by deleting the edge $\langle \emptyset, 0\rangle$.

Let $(t,z) \in V(\mathbb{L})$. We construct a (random) set of vertices of $\mathbb{T}^+_m(t,z)$ in the following way. We begin by colouring (t,z) *red*. Next we examine vertices of $\mathbb{T}^+_m(t,z)$ of the form (u,z) where $\delta_{\mathbb{T}}(t,u) = n$. There is a unique path $\pi_z(t,u)$ of $\mathbb{T} \times \{z\}$ joining (t,z) to (u,z); with this path we associate a 'strip'

$$S_z(t,u) = \{(s,y) \in V(\mathbb{L}) : (s,z) \in \pi_z(t,u), |y-z| \leq m\}$$

together with all associated edges of $\mathbb{L}$ with at least one endvertex of the form (s,y) with $s \neq t,u$ and $|y-z| \leq m$. We colour the vertex (u,z) *red* if and only if $(t,z) \leftrightarrow (u,z)$ in $S_z(t,u)$. Having coloured the vertices (u,z) with $\delta_{\mathbb{T}}(t,u) = n$, we turn to those vertices (w,z) of $\mathbb{T}^+_m(t,z)$ with $\delta_{\mathbb{T}}(t,w) = 2n$. Let (w,z) be such a vertex. There exists a unique vertex u of $\mathbb{T}$ with $\delta_{\mathbb{T}}(t,u) = \delta_{\mathbb{T}}(u,w) = n$. We colour (w,z) *red* if and only if (a) (u,z) is red, and (b) $(u,z) \leftrightarrow (w,z)$ in the strip $S_z(u,w)$. We proceed inductively to obtain a set of red vertices. Clearly the set of red vertices is the set of members of a branching process with mean family-size $k^n p_m(n)$, and we denote this set by $T(t,z)$.

We say that $T(t,z)$ and $T(t,y)$ *overlap infinitely often* (i.o.) if there exist infinitely many vertices $w \in \mathbb{T}^+(t)$ such that (w,z) is red and (w,y) is red. Let $0 \leq \epsilon < 1$. We call the (random) set $T(t,z)$ *ϵ-robust* if (conditional on $T(t,z)$) the probability that $T_1 = T(t,z)$ and $T_2 = T(t,z+2m+1)$ overlap i.o. is strictly larger than ϵ. We note that the (unconditional) probability that T_1 and T_2 overlap i.o. is exactly the probability that the set of vertices w of $\mathbb{T}^+(t)$ such that both (w, z) and $(w, z + 2m + 1)$ are red is infinite. Thus this probability equals the probability that a branching process with mean family-size $k^n p_m(n)^2$ is infinite; such a process is supercritical by (4.8), and therefore the probability in question is strictly positive. Thus

$$\begin{aligned} 0 &< P_{\tau,\lambda}(T_1 \text{ and } T_2 \text{ overlap i.o.}) \\ &= E_{\tau,\lambda}\big(P_{\tau,\lambda}(T_2 \text{ overlaps } T_1 \text{ i.o.} | T_1)\big) \\ &\le P_{\tau,\lambda}(T_1 \text{ is } \epsilon\text{-robust}) + \epsilon P_{\tau,\lambda}(T_1 \text{ is 0-robust but not } \epsilon\text{-robust}) \end{aligned}$$

implying that

$$P_{\tau,\lambda}(T_1 \text{ is } \epsilon\text{-robust}) \ge \frac{P_{\tau,\lambda}(T_1 \text{ and } T_2 \text{ overlap i.o.}) \ - \epsilon}{1 - \epsilon},$$

where the final quantity is strictly positive for all sufficiently small non-negative ϵ. Clearly $P_{\tau,\lambda}(T_1 \text{ is } \epsilon\text{-robust})$ is a decreasing function of ϵ, and we claim that

$$P_{\tau,\lambda}(T_1 \text{ is 0-robust}) = P_{\tau,\lambda}(|T_1| = \infty). \tag{4.9}$$

Certainly the left-hand side is no larger than the right-hand side; to prove equality it suffices to show that

$$P_{\tau,\lambda}(T_1 \text{ is 0-robust}) \ge P_{\tau,\lambda}(|T_1| = \infty). \tag{4.10}$$

To see this we argue as follows. Let ϵ be small and positive. We grow T_1 generation by generation. Each time we reach a new red vertex, there is a strictly positive probability that this vertex is the root of an ϵ-robust tree in future generations. If T_1 is infinite than a.s. we encounter such a red vertex at some stage. If N is the generation number of the first such vertex reached, then there is probability at least $\tau^{nN}\epsilon$ (> 0), that T_2 overlaps T_1 i.o. For any given T_1, we write $\rho(T_1)$ for the supremum of the values of ϵ for which T_1 is ϵ-robust, with the convention that $\rho(T_1) = -1$ if T_1 is either finite or not 0-robust. We have proved that

$$P_{\tau,\lambda}(\rho(T_1) > 0) = P_{\tau,\lambda}(T_1 \text{ is infinite}). \tag{4.11}$$

Calculations related to these may be found in Lyons (1988).

Having set the scene, we move on to the proof proper. Let (t, z) be a vertex of $\mathbb{L}$ which is in an infinite open cluster, say $C(t, z)$, of $\mathbb{L}$. We claim that $C(t, z)$ contains a.s. some vertex of $\mathbb{T} \times \{y\}$ for infinitely many values of y $(\in \mathbb{Z})$. Suppose to the contrary that $C(t, z)$ is confined to some layer $\mathbb{T} \times \{M, \dots, N\}$. Then there exists some positive integer I which is maximal with the property that $C(t,z)$ contains infinitely many vertices of $\mathbb{T} \times \{I\}$. By an argument similar to that in the proof of part (i) of Lemma 3, this event has probability 0 for any given value I, so that a.s. no such I exists. Thus $C(t,z)$ contains a.s. vertices of $\mathbb{T} \times \{y\}$ for every value of y. Growing $C(t,z)$ in the usual algorithmic way (see for example Aizenman, Kesten, and Newman 1987 or Grimmett 1989), we find that $C(t,z)$ contains a.s. some vertex (u,y) which is the root of a robust (i.e. 0-robust) $T(u,y)$ in $\mathbb{T}_m^+(u,y)$ (in fact this will hold for infinitely many values of y); this holds since, each time the growth process reaches a new plane $\mathbb{T} \times \{y\}$, arriving from $\mathbb{T} \times \{y-1\}$ say, there is a strictly positive probability that the hitting point (v,y) is joined to $(v, y+m)$ by a direct path of open edges and in addition $(v, y+m)$ is the root of a robust $T(v, y+m)$ in $\mathbb{T}_m^+(v, y+m)$. By a similar argument we may (and will) assume that $\delta_{\mathbb{T}}(u, \emptyset)$ is a multiple of n.

Suppose now that (s,y) and (t,z) are distinct vertices of $\mathbb{L}$ which are in infinite open clusters. We wish to show that $(s,y) \leftrightarrow (t,z)$ a.s. on this event. Almost surely, $C(s,y)$ and $C(t,z)$ contain vertices (a,i) and (b,j) (respectively) which are the roots of robust sets $T(a,i)$ and $T(b,j)$ in $\mathbb{T}_m^+(a,i)$ and $\mathbb{T}_m^+(b,j)$ respectively (and with $\delta_{\mathbb{T}}(a,\emptyset)$ and $\delta_{\mathbb{T}}(b,\emptyset)$ multiples of n, and, if desired, with $|i-j| > 2m$). It suffices therefore to show that there is probability 0 that there exist two such distinct vertices (a,i) and (b,j) which are the roots of such robust sets but which are not connected by an open path of $\mathbb{L}$. Let (a,i) and (b,j) be distinct vertices of $\mathbb{L}$. We say that (a,i) is *related* to (b,j) if $V(\mathbb{T}^+(a)) \cap V(\mathbb{T}^+(b)) \neq \emptyset$, and *unrelated* otherwise. Suppose first that (a,i) and (b,j) are unrelated, and let $T_1 = T(a,i)$ and $T_2 = T(b,j)$. There exists a shortest path π of $\mathbb{L}$ from (a,i) to (b,j) using no edges of $\mathbb{T}^+(a,i)$ or $\mathbb{T}^+(b,j)$ and which is open with probability $\lambda^{|i-j|}\tau^{\delta_{\mathbb{T}}(a,b)} = \sigma$, say. If T_1 is infinite then it is a.s. ϵ-robust for some (random) $\epsilon > 0$ (any ϵ in $(0, \rho(T_1))$ will do) by (4.11). Consider the graphs $\mathbb{T}_m^+(a, i + k(2m+1))$ as k ranges over the positive integers. Conditional on T_1, we have that if $|T_1| = \infty$ then each vertex $(a, i + k(2m+1))$ has a strictly positive probability (depending on T_1) of being the root of a 'red' branching process in $\mathbb{T}_m^+(a, i + k(2m+1))$ which overlaps T_1 i.o., and furthermore the corresponding events are independent for different values of k. On the event that $(a, i+k(2m+1))$ is such a vertex, it is the case that $(a, i+k(2m+1)) \leftrightarrow (a,i)$ a.s., since there is (conditional) probability 1 that, given two 'red' processes which overlap i.o., we may find two points, one from each process, which lie in the same copy of $\mathbb{Z}$ and with the property that the path of $\mathbb{Z}$-edges joining them is open. (We are using here the fact that the strips $S_z(t,u)$ did not include $\mathbb{Z}$-edges along $\{t\} \times \mathbb{Z}$ or $\{u\} \times \mathbb{Z}$.) Let A_k be the event that (i) $(a, i+k(2m+1))$ is the root of a red tree which overlaps T_1 i.o., (ii) $(b, j+k(2m+1))$ is the root of a red tree which overlaps

T_2 i.o., and (iii) the path π_k with vertex set $\{(t, z+k(2m+1) : (t,z) \in \pi\}$ is open. Conditional on T_1 and T_2, the events $\{A_k : k \geq 1\}$ are independent and each has probability at least $\frac{1}{2}\rho(T_1)\rho(T_2)\sigma$ (> 0); hence, a.s. some A_k occurs, so that $(a,i) \leftrightarrow (b,j)$ a.s. by the remarks above. Therefore

$$\begin{aligned} P_{\tau,\lambda}\big((a,i) \leftrightarrow (b,j);\ |T_1| = |T_2| = \infty\big) \\ &= \iint\limits_{T_1,T_2:|T_1|=|T_2|=\infty} dP_{\tau,\lambda}(T_1)dP_{\tau,\lambda}(T_2)P_{\tau,\lambda}\big((a,i) \leftrightarrow (b,j)|T_1,T_2\big) \\ &= \iint\limits_{T_1,T_2:|T_1|=|T_2|=\infty} dP_{\tau,\lambda}(T_1)dP_{\tau,\lambda}(T_2) \\ &= P_{\tau,\lambda}(|T_1| = \infty, |T_2| = \infty) \end{aligned}$$

as required.

Suppose finally that (a,i) and (b,j) are related and are the roots of infinite red processes $T(a,i)$ and $T(b,j)$, respectively. Suppose also that $\delta_{\mathbb{T}}(a,\emptyset)$ and $\delta_{\mathbb{T}}(b,\emptyset)$ are multiples of n, and that $|i-j| > 2m$. If $T(a,i)$ and $T(b,j)$ overlap i.o., then $(a,i) \leftrightarrow (b,j)$ a.s. by an earlier argument, and so it suffices to assume that $T(a,i)$ and $T(b,j)$ do not overlap i.o. In this case there exists a (random) positive integer R such that $T(a,i)$ and $T(b,j)$ contain *no* overlaps in the set $S \times \mathbb{Z}$ where $S = V(\mathbb{T}^+(a)) \cap V(\mathbb{T}^+(b)) \cap \{t : \delta_{\mathbb{T}}(a,t) \geq R\}$. We may pick c, $d \in S$ such that (c,i) and (d,j) are the roots of infinite red processes in $\mathbb{T}^+_m(c,i)$ and $\mathbb{T}^+_m(d,j)$ and such that (c,i) and (d,j) are unrelated. The chance that such (c,i) and (d,j) are in different infinite open clusters of $\mathbb{L}$ is 0, by the preceding argument, and the proof is complete. ∎

5. Percolation in $\infty + d$ Dimensions

In this section we consider the lattice $\mathbb{L}_d = \mathbb{T} \times \mathbb{Z}^d$ for $d > 1$ and discuss briefly how the arguments and results differ from the case $d = 1$. We continue to denote vertices in $\mathbb{Z}^d$ by z, z_1, and so on.

To modify the analysis which led to Propositions 1 and 4, we note that $(\emptyset, 0) \leftrightarrow (t,z)$ in $\mathbb{L}_d$ if and only if for some n there is a route $\mathbf{t} = (t_0, t_1, \dots, t_n)$ in $\mathbb{T}$ from $\emptyset$ to t and a sequence $\mathbf{z} = (z_{-1} = 0, z_0, z_1, \dots, z_n = z)$ such that:

(a) $\langle (t_{i-1}, z_{i-1}), (t_i, z_{i-1}) \rangle$ is open, for $i = 1, \dots, n$,

(b) $(t_i, z_{i-1}) \leftrightarrow (t_i, z_i)$ in $\{t_i\} \times \mathbb{Z}^d$, for $i = 0, \dots, n$,

(c) if $t_i = t_j$, then $(t_i, z_i) \not\leftrightarrow (t_j, z_j)$ in $\{t_i\} \times \mathbb{Z}^d$, for $0 \leq i < j \leq n$.

Condition (c) is a 'cluster self-avoiding' property; it implies among other things that (as in the $d = 1$ case) $z_{i-1} \neq z_{i-2}$ when $t_i = t_{i-2}$. By successively conditioning on the $\{t_i\} \times \mathbb{Z}^d$ clusters of (t_i, z_{i-1}), one sees that inequality (3.10) remains valid, but with $\mathcal{X}_\lambda$ replaced by $\mathcal{X}_{\lambda,d}$, the expected cluster size for standard bond percolation on $\mathbb{Z}^d$ with isotropic bond density λ. The remainder of the analysis remains valid and leads to the following extension of Propositions 1 and 4.

PROPOSITION 6. *If*

$$\tau k \mathcal{X}_{\lambda,d}\big(1+\sqrt{1-\mathcal{X}_{\lambda,d}^{-1}}\big) < 1 \tag{5.1}$$

then there is a.s. no infinite open cluster in $\mathbb{L}_d$. *If* $\theta_{\mathbb{L}_d}(\tau,\lambda) > 0$ *but*

$$\tau\sqrt{k}\mathcal{X}_{\lambda,d}\big(1+\sqrt{1-\mathcal{X}_{\lambda,d}^{-1}}\big) < 1, \tag{5.2}$$

then there exists a.s. infinitely many infinite open clusters in $\mathbb{L}_d$.

Proposition 6 provides lower bounds $\tau = \underline{\tau}_{l,d}(\lambda)$ and $\tau = \underline{\tau}_{u,d}(\lambda)$ for the lower and upper curves of Figure 2. These curves are only implicitly defined since they are expressed in terms of $\mathcal{X}_{\lambda,d}$. As $\lambda \to 0$, $\mathcal{X}_{\lambda,d} = 1 + 2d\lambda + o(\lambda)$ so that, just as when $d = 1$,

$$\underline{\tau}_{l,d}(0) = \frac{1}{k},\ \ \underline{\tau}'_{l,d}(0) = -\infty;\ \ \underline{\tau}_{u,d}(0) = \frac{1}{\sqrt{k}},\ \ \underline{\tau}'_{u,d}(0) = -\infty. \tag{5.3}$$

On the other hand, for $d > 1$, the critical probability $\lambda_c(d)$ for percolation satisfies $\lambda_c(d) < 1$ and $\mathcal{X}_{\lambda,d}$ diverges as $\lambda \uparrow \lambda_c$ (Menshikov 1986; Menshikov, Molchanov, and Sidorenko 1986; Aizenman and Barsky 1987). It follows that

$$\underline{\tau}_{l,d}(\lambda) \sim \frac{1}{2k}\mathcal{X}_{\lambda,d}^{-1},\ \ \underline{\tau}_{u,d}(\lambda) \sim \frac{1}{2\sqrt{k}}\mathcal{X}_{\lambda,d}^{-1}\ \text{ as }\ \lambda \uparrow \lambda_c(d), \tag{5.4}$$

so that $\underline{\tau}_{l,d}(\lambda_c(d)) = 0 = \underline{\tau}_{u,d}(\lambda_c(d))$. Since $\mathcal{X}_{\lambda,d}^{-1} = O(\lambda_c(d) - \lambda)$ as $\lambda \uparrow \lambda_c(d)$ (Aizenman and Newman 1984), we see that the derivatives (with respect to λ) of the two lower curves are finite for any d; however since for $d < 6$ it is expected that $\mathcal{X}_{\lambda,d}^{-1}$ behaves as $(\lambda_c(d) - \lambda)^\gamma$ with critical exponent $\gamma > 1$, these derivatives should be zero at $\lambda_c(d)$. These derivatives have recently been proved to be non-zero in sufficiently high dimensions (see Hara and Slade 1989a,b).

The analysis which led to Propositions 2 and 5 extends almost unchanged to the cases of two and more dimensions. We consider anisotropic bond percolation on $\mathbb{Z} \times \mathbb{Z}^d$ with edge density τ for edges in the first ($\mathbb{Z}$) component and λ for $\mathbb{Z}^d$-edges, and we define $\phi_m^d(\tau,\lambda)$ exactly as in (2.2) with

$$T_m(n) = \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}^d : 0 \le z_1 \le n, z_2 \in [-m, m]^d\}$$

(once again without the edges in its 'left' and 'right' boundary faces). Then the (horizontal) correlation length ϕ^d is given by

$$\begin{aligned}\phi^d(\tau,\lambda) &= \lim_{m\to\infty} \phi^d_m(\tau,\lambda)\\ &= \lim_{n\to\infty}\left\{-\frac{1}{n}\log P\big((0,0)\leftrightarrow(n,0)\text{ in }\mathbb{Z}\times\mathbb{Z}^d\big)\right\}\\ &= \lim_{n\to\infty}\left\{-\frac{1}{n}\log P\big((0,0)\leftrightarrow\{n\}\times\mathbb{Z}^d\text{ in }[0,\infty)\times\mathbb{Z}^d\big)\right\}.\end{aligned}$$

PROPOSITION 7. *If*

$$ke^{-\phi^d(\tau,\lambda)} > 1, \tag{5.5}$$

then there is a.s. an infinite open cluster in $\mathbb{L}_d$. *If*

$$ke^{-2\phi^d(\tau,\lambda)} > 1, \tag{5.6}$$

then there is a.s. a unique infinite open cluster in $\mathbb{L}_d$.

Proposition 7 provides implicitly defined upper bounds $\tau = \overline{\tau}_{l,d}(\lambda)$ and $\tau = \overline{\tau}_{u,d}(\lambda)$ for the lower and upper curves of Figure 2. Since $e^{-\phi^d} \geq \tau$ (recall (2.10)), one knows that

$$\overline{\tau}_{l,d}(0) = 1/k, \quad \overline{\tau}_{u,d}(0) = 1/\sqrt{k}. \tag{5.7}$$

The generalization of (2.11) is

$$e^{-\phi^d(\tau,\lambda)} \geq 1 - E\big((1-\tau)^{|C_\lambda|}\big) \tag{5.8}$$

where $|C_\lambda|$ is the cluster size at the origin of isotropic bond percolation on $\mathbb{Z}^d$ with edge-density λ. Inequality (5.8) can be used to estimate the slopes of $\overline{\tau}_{l,d}$ and $\overline{\tau}_{u,d}$ at $\lambda = 0$, but it does not provide the correct values of $\overline{\tau}_{l,d}(\lambda)$ and $\overline{\tau}_{u,d}(\lambda)$ at $\lambda = \lambda_c(d)$, since if $|C_{\lambda_c(d)}| < \infty$ a.s. (as is known for $d = 2$ and presumed for all $d > 2$; see Barsky, Grimmett, and Newman 1989 for the corresponding result for half-spaces, and Hara and Slade 1989a,b for the full-space result in high dimensions) the right hand side of (5.8) cannot be made larger than $1/k$ (or $1/\sqrt{k}$) as $\lambda \uparrow \lambda_c(d)$ unless τ is bounded away from zero. To see that

$$\overline{\tau}_{l,d}(\lambda_c(d)) = 0, \quad \overline{\tau}_{u,d}(\lambda_c(d)) = 0, \tag{5.9}$$

simply note that $\phi^d(\tau,\lambda) = 0$ if (τ,λ) is such that there is percolation in $\mathbb{Z}\times\mathbb{Z}^d$, and this is easily seen to occur for any small τ if λ is sufficiently close to $\lambda_c(d)$. Careful versions of such arguments show that λ need be no closer than some multiple of τ as $\tau \downarrow 0$ which implies finite bounds for the derivatives (with respect to λ) of $\overline{\tau}_{l,d}$ and $\overline{\tau}_{u,d}$ at $\lambda = \lambda_c(d)$.

Unfortunately, because of the behaviour of $\mathcal{X}_{\lambda,d}$ discussed previously, we cannot combine our present knowledge about $\overline{\tau}_{l,d}$ and $\underline{\tau}_{u,d}$ near $\lambda = \lambda_c(d)$ to conclude that the region of infinitely many infinite open clusters extends all the way to $(\tau, \lambda) = (0, \lambda_c(d))$. The best we can say for $d \geq 2$ is that for any $k \geq 2$ there *is* such a region (since $\overline{\tau}_{l,d} < \underline{\tau}_{u,d}$ near $\lambda = 0$) and that this region certainly approaches $(\tau, \lambda) = (0, \lambda_c(d))$ as $k \to \infty$. This last fact follows by taking a fixed value of λ near to $\lambda_c(d)$, and combining the inequalities

$$\underline{\tau}_{u,d}(\lambda) \geq \frac{1}{2\mathcal{X}_{\lambda,d}\sqrt{k}}$$

and

$$\overline{\tau}_{l,d}(\lambda) \leq \tau_0 = \frac{1}{k\mathcal{X}_{\lambda,d}} + O(k^{-2}) \quad \text{as} \quad k \to \infty$$

where τ_0 is the root of the equation

$$1 - E\big((1-\tau)^{|C_\lambda|}\big) = \frac{1}{k}.$$

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