CHAPTER 2
Migration Processes

In this chapter we shall meet some of the simpler systems in which customers (or individuals) move about between a number of queues (or colonies). First we shall consider further the simple queue introduced in Section 1.3.

2.1 THE OUTPUT FROM A SIMPLE QUEUE

In Section 1.3 it was shown that if $n(t)$ is the number of customers in an $M/M/1$ queue at time t then in equilibrium $n(t)$ is a reversible Markov process. A typical realization of $n(t)$ is illustrated in Fig. 2.1. Note that the points in time at which $n(t)$ jumps upwards form a Poisson process of rate ν since these points correspond to arrivals at the queue. Now $n(t)$ is reversible and hence the points in time at which $n(-t)$ jumps upwards must also form a Poisson process of rate ν. But if $n(-t)$ jumps upwards at time $-t_0$ then $n(t)$ jumps downwards at time t_0, and so the points in time at which $n(t)$ jumps downwards must form a Poisson process of rate ν. But these points correspond to departures from the queue. We have thus shown that in equilibrium the points in time at which customers leave the queue (the departure process) form a Poisson process of rate ν. The line of argument can be used to establish a little more. Let t_0 be a fixed instant in time. Since $n(t)$ is reversible, the departure process up until time t_0 and the number in the queue at time t_0 have the same joint distribution as the arrival process after time $-t_0$ and the number in the queue at time $-t_0$. But the arrival process after time $-t_0$ is independent of the number in the queue at time $-t_0$, and hence the departure process prior to time t_0 is independent of the number in the queue at time t_0. The next theorem summarizes these results.

Theorem 2.1. *In equilibrium the departure process from an $M/M/1$ queue is a Poisson process, and the number in the queue at time t_0 is independent of the departure process prior to time t_0.*

In some ways this result is surprising, since while the server is busy departures occur at rate μ and while the server is idle departures occur at rate zero. It is difficult, however, to analyse the departure process using this approach since the length of a busy period and the departure process during this period are not independent. The dependence is such that if we observe

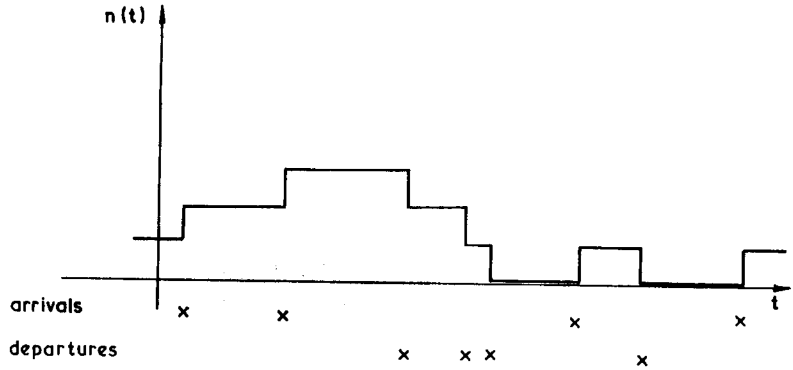

Fig. 2.1 A realization of the process $n(t)$

the entire departure process from an $M/M/1$ queue for $-\infty < t < \infty$, but know nothing of the times of arrival or the numbers in the queue, then we can determine the arrival rate ν but can learn nothing of the service rate μ.

The reasoning which led to Theorem 2.1 will apply to any queue with a Poisson arrival process for which the number in the queue is a birth and death process, for example the $M/M/s$ queue (Exercise 1.3.3). More generally it will apply whenever a queue with a Poisson arrival process can be represented by a reversible Markov process, provided an arrival causes the process to change state and the reverse transition corresponds to a departure. A further example of such a queue is the two-server queue discussed in Section 1.5. It occasionally requires some guile to find an appropriate process, as the following example illustrates.

A telephone exchange. Consider the model of a telephone exchange with K lines described in Section 1.3. The number of calls in progress at time t, n, is a reversible Markov process, but one which does not always change state when a call is initiated. Consider, however, the process (n, f) where the flip-flop variable f takes the value zero or unity and changes value whenever a call is lost. Clearly this process changes state whenever a call is initiated, and it is easily checked that the process is reversible with equilibrium distribution

$$\pi(n, f) = \tfrac{1}{2}\pi(n) \qquad n = 0, 1, \ldots, K; \quad f = 0, 1$$

where $\pi(n)$ is the equilibrium distribution of the process n. Moreover, transitions of the process associated with the completion of a call or the loss of a call are just the reverse transitions of those associated with the initiation of a call. Thus the points in time at which a call is lost or is completed form a Poisson process. If the points in time at which a call is lost are considered alone they form a more complicated point process, but one which is reversible (Exercise 2.1.3).

A two-server queue. Consider now the two-server queue introduced in Section 1.7. The Markov process representing this queue (Fig. 1.4a) is not reversible. Nevertheless, we do know the form of the reversed process (Fig. 1.4b). Indeed the reversed process can be regarded as representing an identical two-server queue but with a different interpretation being given to states $0A$ and $0B$ (Exercise 1.7.5). Observe that if a transition in the reversed process corresponds to an arrival then the reverse transition in the original process corresponds to a departure. Arrivals at the queue represented by the reversed process form a Poisson process and the arrival process at this queue after time $-t_0$ is independent of the state of the reversed process at time $-t_0$. Hence departures from the queue represented by the original process form a Poisson process and the state of the original process at time t_0 is independent of the departure process prior to time t_0. This example shows that it is not reversibility as such that leads to the results, but rather the particular form of the reversed process.

Exercises 2.1

1. Consider a queue with s identical servers who each take an exponentially distributed amount of time to serve a customer. Suppose that an arriving customer leaves immediately without being served (he *balks*), with a probability depending on the number in the queue, and that if he does join the queue he gives up and defects after an exponentially distributed amount of time unless his service has begun beforehand. Use both of the following approaches to show that if the arrival process is Poisson then in equilibrium the departure process is Poisson, provided all departing customers are counted.
   (i) Represent the queue by a Markov process (n, f) as in the telephone exchange model.
   (ii) Approximate the queue by one at which customers who decide on arrival that they will leave without service remain in the queue for an exponentially distributed time with mean ξ^{-1} where ξ is very large. Let m be the number of such customers in the queue. Suppose that while m is positive service and defection are suspended and further arrivals decide to leave the queue without service, i.e. they increase m. Let n be the number of other customers in the queue. Find the equilibrium distribution of the Markov process (n, m).
2. Show that the departure process from the queue considered in the preceding exercise remains Poisson if the defection rate of a customer depends upon how many are in front of him in the queue. Show that the departure processes from the many-server queues considered in Exercises 1.5.6, 1.5.7, and 1.7.6 are Poisson and remain so even if customers may balk or defect.

3. Before we can assert that a point process is reversible we need to characterize a point process in a form to which our definition of reversibility (Section 1.2) can apply. The simplest way to do this is with a flip-flop variable $f(t)$ defined as follows: $f(t)$ takes the value zero or unity and changes value at the points in time of the point process. Call the point process reversible if $f(t)$ is reversible. Show that in the model of a telephone exchange the intervals between successive lost calls are independent, and deduce that the points in time at which a call is lost form a reversible point process. (A point process in which the intervals between successive points are independent is called a renewal process.) Observe that a stationary renewal process is always reversible.
4. In the model of a telephone exchange show that the points in time at which a call is completed form a point process which when reversed in time is statistically indistinguishable from that formed by the points in time at which a call is successfully connected. Show that without the time reversal the two processes will differ unless $K=1$.

2.2 A SERIES OF SIMPLE QUEUES

The most obvious application of Theorem 2.1 is to a series of J single-server queues arranged so that when a customer leaves a queue he joins the next one, until he has passed through all queues (Fig. 2.2). Suppose the arrival stream at queue 1 is Poisson at rate ν and that service times at queue j are exponentially distributed with mean μ_j^{-1}, where $\nu<\mu_j$ for $j=1,2,\ldots,J$. Suppose further that service times are independent of each other, including those of the same customer in different queues, and of the arrival stream at queue 1. Let $n_j(t)$ be the number of customers in queue j at time t. Queue 1 viewed in isolation is simply an $M/M/1$ queue and hence the departure process from it is Poisson, by Theorem 2.1. Thus the arrival process at queue 2 is Poisson, and so it, too, viewed in isolation, is an $M/M/1$ queue. Proceeding with this argument we see that queue j viewed in isolation is an $M/M/1$ queue, and hence in equilibrium

$$\pi_j(n_j)=\left(1-\frac{\nu}{\mu_j}\right)\left(\frac{\nu}{\mu_j}\right)^{n_j}$$

What is not yet clear is the joint distribution of $(n_1, n_2, \ldots, n_J)$. Now Theorem 2.1 also states that $n_1(t_0)$ is independent of the departure process from queue 1 prior to t_0. But $(n_2(t_0), n_3(t_0), \ldots, n_J(t_0))$ is determined by the

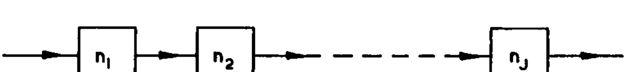

Fig. 2.2 A series of queues

departure process from queue 1 prior to t_0 and service times at queues $2, 3, \ldots, J$. Hence $n_1(t_0)$ is independent of $(n_2(t_0), n_3(t_0), \ldots, n_J(t_0))$. Similarly, $n_j(t_0)$ is independent of $(n_{j+1}(t_0), \ldots, n_J(t_0))$. Thus $n_1(t_0), n_2(t_0), \ldots, n_J(t_0)$ are mutually independent, and so in equilibrium

$$\pi(n_1, n_2, \ldots, n_J) = \prod_{j=1}^{J} \left(1 - \frac{\nu}{\mu_j}\right)\left(\frac{\nu}{\mu_j}\right)^{n_j}$$

The above approach is clearly of much wider applicability. The queues in the system can be of any of the forms discussed in the last section, and indeed the final queue need not be restricted even in this way. It is not essential that customers who leave queue j should join queue $j+1$; they may leave the system or jump to a queue between $j+1$ and J. We shall not pursue this approach, however, since it breaks down when a customer leaving queue j is allowed to jump back to a queue between 1 and j. Such behaviour will be discussed in the following sections.

Consider now the experience of an individual customer as he passes through the series of J simple queues described at the beginning of this section.

Theorem 2.2. *If the discipline at each queue in a series of J simple queues is first come first served, then in equilibrium the waiting times of a customer at each of the J queues are independent.*

Proof. The first step of the proof is to establish that in equilibrium the waiting time of a customer at a first come first served $M/M/1$ queue is independent of the departure process prior to his departure. Let $n(t)$ be the number of customers in the queue at time t. Then $n(-t)$ can also be regarded as the number in a first come first served $M/M/1$ queue at time t, since its behaviour is statistically indistinguishable from that of $n(t)$. Now if a customer arrives at the original queue at time t_0 and leaves at time t_1 then $n(-t)$ will signal the arrival of a customer at time $-t_1$ and the departure of the *same* customer at time $-t_0$. But the waiting time of this customer is independent of the arrivals signalled by $n(-t)$ after time $-t_1$. Hence in the original queue the departure process prior to time t_1 is independent of the waiting time of the customer who leaves at time t_1.

Consider a customer leaving queue 1. Customers who leave queue 1 after him cannot reach any subsequent queue before him: the queue discipline and the assumption of a single server at the next $J-2$ queues ensure this. Now his waiting time at queue 1 is independent of the arrival process at queue 2 prior to his arrival, and hence is independent of his waiting time at queues $2, 3, \ldots, J$. Similarly, his waiting time at queue j is independent of his waiting times at queues $j+1$, $j+2, \ldots, J$, and hence the theorem is proved.

It is clear from the proof of Theorem 2.2 that the final queue in the system is not required to be simple. For example waiting times would still be independent if the Jth queue were a first come first served $M/G/s$ queue, i.e. an s-server queue at which service times have a general distribution. Few other generalizations are possible; the independence of waiting times is a much less common result than the independence of queue sizes.

Exercises 2.2

1. If in a series of simple queues $\mu_1 = \mu_2 = \cdots = \mu_J$ show that the Markov process $(n_1, n_2, \ldots, n_J)$ is dynamically reversible.
2. Observe that in a series of simple queues the waiting time of a customer at queue j is exponential with mean $(\mu_j - \nu)^{-1}$. Deduce that the time taken for a customer to pass through the system is the sum of J independent exponentially distributed random variables, and has mean $\sum_j (\mu_j - \nu)^{-1}$ and variance $\sum_j (\mu_j - \nu)^{-2}$.
3. Consider two stacks, as described in Exercise 1.3.8, arranged so that customers leaving the first stack join the second. Show that in equilibrium the waiting time of a customer at the first stack is independent of the departure process *subsequent* to his departure. Deduce that the waiting times of a customer at the two stacks are independent.
4. Let $n(t)$ be the number of customers in an $M/M/s$ queue at time t. Suppose the queue discipline is first come first served, and let t_0 and t_1 be points in time at which $n(t)$ increases and decreases respectively. From the realization $n(t)$, $-\infty < t < \infty$, the probability P that the customer arriving at time t_0 is the one leaving at time t_1 can be calculated. Note that P will be zero or unity if $s = 1$. If the reversed process $n(-t)$ is regarded as representing the number in a first come first served $M/M/s$ queue, show that P is the probability that in this queue the customer who arrives at time $-t_1$ is the one who leaves at time $-t_0$. Deduce that in equilibrium the waiting time of a customer at a first come first served $M/M/s$ queue is independent of the departure process prior to his departure.
5. Consider a series of J first come first served $M/M/s$ queues in equilibrium. Let s_j be the number of servers at queue j. Deduce from the previous exercise that the waiting times of a customer at two successive queues are independent. Consider the case $J \geq 3$, $s_1 = s_3 = 1$, $s_2 = \infty$, $\mu_1 = \mu_2 = \mu_3$. Show that if a customer's waiting time at queue 1 is large then the probability that the customer entering queue 1 after him will overtake him and be present in queue 3 when he arrives there is close to one-eighth. Deduce that although a customer's waiting times at queues 1 and 2 or at queues 2 and 3 are independent, his waiting times at queues 1 and 3 are dependent. Deduce from the previous exercise that if $s_j = 1$

unless $j=1$ or J then the waiting times of a customer at each of the J queues are independent.

6. Consider a series of two simple queues in equilibrium. Suppose that an arriving customer finds queue 1 empty. Show that the probability queue 2 will be empty when he reaches it is

$$1-\frac{\nu}{\mu_2}+\frac{\nu}{\mu_2}\left(\frac{\mu_2-\nu}{\mu_1+\mu_2-\nu}\right)$$

Deduce that although a customer's waiting times at the two queues are independent his queueing times are not.

2.3 CLOSED MIGRATION PROCESSES

The elegant but delicate method of analysis used in the preceding sections breaks down if customers can rejoin queue 1 after leaving queue J. In this and the next section we shall use an alternative approach which can deal with such behaviour. The approach readily yields equilibrium distributions for the number of customers in each queue, but is not as informative about the time taken by a customer to pass through a sequence of queues.

We shall call the model to be examined a migration process. The main applications are to queueing rather than to biological systems, but the idea of individuals moving between colonies makes exposition easier and the alternative term 'queueing network' seems more appropriate for the model of the next chapter. In this section we shall consider a closed migration process where individuals cannot enter or leave the system but can only move between colonies. Thus the total number of individuals in the system, N, is fixed.

Consider a set of J colonies and let n_j denote the number of individuals in colony j, for $j=1,2,\ldots,J$. Define an operator T_{jk} to act upon the vector $\mathbf{n}=(n_1,n_2,\ldots,n_J)$ as follows:

$$T_{jk}(n_1,n_2,\ldots,n_j,\ldots,n_k,\ldots,n_J)=(n_1,n_2,\ldots,n_j-1,\ldots,n_k+1,\ldots,n_J)$$

if $j<k$. Similarly,

$$T_{jk}(n_1,n_2,\ldots,n_k,\ldots,n_j,\ldots,n_J)=(n_1,n_2,\ldots,n_k+1,\ldots,n_j-1,\ldots,n_J)$$

if $k<j$. Thus T_{jk} moves an individual from colony j to k. We shall study $\mathbf{n}$ under the assumption that it is a Markov process with transition rates given by

$$q(\mathbf{n},T_{jk}\mathbf{n})=\lambda_{jk}\phi_j(n_j) \tag{2.1}$$

where $\phi_j(0)=0$ and for simplicity $\lambda_{jj}=0$. The parameter λ_{jk} can be viewed as measuring the intrinsic tendency for movement from colony j to colony k; the function $\phi_j(n)$ then measures the extent to which this tendency is affected by the number of individuals in colony j. To ensure that $\mathbf{n}$ is irreducible within the state space

$$\mathcal{S}=\left\{\mathbf{n} \mid n_j \geq 0, j=1,2,\ldots,J;\ \sum_{j=1}^{J} n_j = N\right\}$$

we shall require that $\phi_j(n)>0$ if $n>0$ and that the parameters λ_{jk} allow an individual to pass between any two colonies, either directly or indirectly via a chain of other colonies. We shall call the process $\mathbf{n}$ a closed migration process.

As an example of the behaviour transition rates (2.1) can allow consider the special case

$$\phi_j(n)=\min(n, s)$$

With this function colony j behaves as a queue with s servers at which service times are exponentially distributed with mean λ_j^{-1}, where

$$\lambda_j = \sum_k \lambda_{jk}$$

An individual departing from this queue joins colony k with probability λ_{jk}/λ_j.

If $\phi_j(n)=n$ for all j then the migration process is called linear, and the individuals can be considered to be moving independently of one another. If $N=1$ then the single individual in the system performs a random walk on the set of colonies, and if $\alpha_1, \alpha_2, \ldots, \alpha_J$ is the unique collection of positive numbers summing to unity which satisfy

$$\alpha_j \sum_k \lambda_{jk} = \sum_k \alpha_k \lambda_{kj} \qquad j=1,2,\ldots,J \tag{2.2}$$

then α_j is the equilibrium probability that the individual is in colony j.

Theorem 2.3. *The equilibrium distribution for a closed migration process is*

$$\pi(\mathbf{n}) = B_N \prod_{j=1}^{J} \frac{\alpha_j^{n_j}}{\prod_{r=1}^{n_j} \phi_j(r)} \qquad \mathbf{n}\in\mathcal{S} \tag{2.3}$$

where B_N is a normalizing constant, chosen so that the distribution sums to unity.

Proof. The equilibrium equations (1.3) are

$$\pi(\mathbf{n}) \sum_{j=1}^{J} \sum_{k=1}^{J} q(\mathbf{n}, T_{jk}\mathbf{n}) = \sum_{j=1}^{J} \sum_{k=1}^{J} \pi(T_{jk}\mathbf{n}) q(T_{jk}\mathbf{n}, \mathbf{n})$$

which become

$$\pi(\mathbf{n}) \sum_{j=1}^{J} \sum_{k=1}^{J} \lambda_{jk}\phi_j(n_j) = \sum_{j=1}^{J} \sum_{k=1}^{J} \pi(T_{jk}\mathbf{n})\lambda_{kj}\phi_k(n_k+1) \tag{2.4}$$

These will be satisfied if we can find a distribution $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, which satisfies

$$\pi(\mathbf{n}) \sum_{k=1}^{J} \lambda_{jk}\phi_j(n_j) = \sum_{k=1}^{J} \pi(T_{jk}\mathbf{n})\lambda_{kj}\phi_k(n_k+1) \tag{2.5}$$

If $n_j = 0$ then, with the convention that $\pi(\mathbf{n})$ vanishes if $\mathbf{n} \notin \mathcal{S}$, equations (2.5) are satisfied trivially. When $n_j > 0$ it is readily verified, using equation (2.2), that the form proposed for $\pi(\mathbf{n})$ satisfies equations (2.5). Thus $\pi(\mathbf{n})$, $\mathbf{n} \in \mathcal{S}$, satisfies the equilibrium equations (2.4) and, since $\mathcal{S}$ is finite, it is clearly possible to choose B_N so that the distribution sums to unity.

The process $\mathbf{n}$ will be reversible if $\alpha_1, \alpha_2, \ldots, \alpha_J$ satisfy

$$\alpha_j\lambda_{jk} = \alpha_k\lambda_{kj}$$

since then the detailed balance conditions

$$\pi(\mathbf{n})\lambda_{jk}\phi_j(n_j) = \pi(T_{jk}\mathbf{n})\lambda_{kj}\phi_k(n_k+1) \tag{2.6}$$

will hold. The relations (2.5) are of a form intermediate between the detailed balance conditions (2.6) and the full balance conditions (2.4). We shall call them partial balance equations. Their connection with the partial balance conditions defined in Exercise 1.6.2 will be explored in Chapter 9; in this chapter our only use of partial balance will be to simplify the verification of equilibrium distributions.

The partial balance equations (2.5) state that in equilibrium the probability flux out of a state due to an individual moving from colony j is equal to the probability flux into that same state due to an individual moving to colony j. This statement is *not* clear a priori, and should be contrasted with the balance equation

$$\sum_{\mathbf{n}\in\mathcal{S}} \pi(\mathbf{n}) \sum_{k=1}^{J} q(\mathbf{n}, T_{jk}\mathbf{n}) = \sum_{\mathbf{n}\in\mathcal{S}} \sum_{k=1}^{J} \pi(T_{jk}\mathbf{n})q(T_{jk}\mathbf{n}, \mathbf{n}) \tag{2.7}$$

which states that in equilibrium the probability flux that an individual leaves colony j is equal to the probability flux that an individual enters colony j. This statement *is* clear a priori (and holds even if the transition rates (2.1) take a more general form) since in equilibrium the mean arrival rate at colony j must equal the mean departure rate from colony j.

Note that if λ_{jk}, $k = 1, 2, \ldots, J$, are decreased by a constant factor and $\phi_j(n)$, $n = 0, 1, 2, \ldots,$ is increased by the same factor, then neither the transition rates (2.1) nor the equilibrium distribution $\pi(\mathbf{n})$ are altered in value. Note also that if the solution $\alpha_1, \alpha_2, \ldots, \alpha_J$ of equations (2.2) is not normalized to sum to unity the expression (2.3) remains valid; the normalizing constant B_N will alter accordingly. These observations can often simplify manipulations, but the task of determining B_N usually remains computationally tedious.

An important class of closed migration processes have the following property:

$$\lambda_{jk} = 0 \qquad \text{unless } k = j+1$$

and

$$\lambda_{j,j+1} = 1$$

for $j = 1, 2, \ldots, J-1$

$$\lambda_{Jk} = 0 \qquad \text{unless } k = 1$$

and

$$\lambda_{J1} = 1$$

Thus an individual repeatedly moves around the cycle of colonies $1, 2, \ldots, J, 1, 2, \ldots$. Such processes are called cyclic queues, and we shall devote the rest of this section to some examples of them.

The provision of spare components. Suppose that there are s_1 machines which each require a certain component in order to operate. A component in use fails after a period which is exponentially distributed with mean ϕ_1^{-1}. It is then replaced from a store of available components unless this is empty, in which case the machine lies idle until a component becomes available. There are s_2 servicing facilities to deal with failed components, and the length of time taken to service a component is exponentially distributed with mean ϕ_2^{-1}. After being serviced a component is returned to the store of available components. There are a total of N components altogether, and an issue of interest is the extent to which increasing N reduces the idle time of the machines.

If we regard the components as customers the system is equivalent to a cyclic queue with

$$\phi_j(n_j) = \phi_j \min(n_j, s_j) \qquad j = 1, 2$$

where n_1 is the number of components in use and in store, and $n_2 = N - n_1$. For a cyclic queue a solution of equation (2.2) is $\alpha_1 = \alpha_2 = \cdots = \alpha_J = 1$ and so Theorem 2.3 shows that the equilibrium distribution is

$$\pi(n_1, n_2) = \frac{B_N}{\prod_{r=1}^{n_1} \phi_1(r) \prod_{r=1}^{n_2} \phi_2(r)}$$

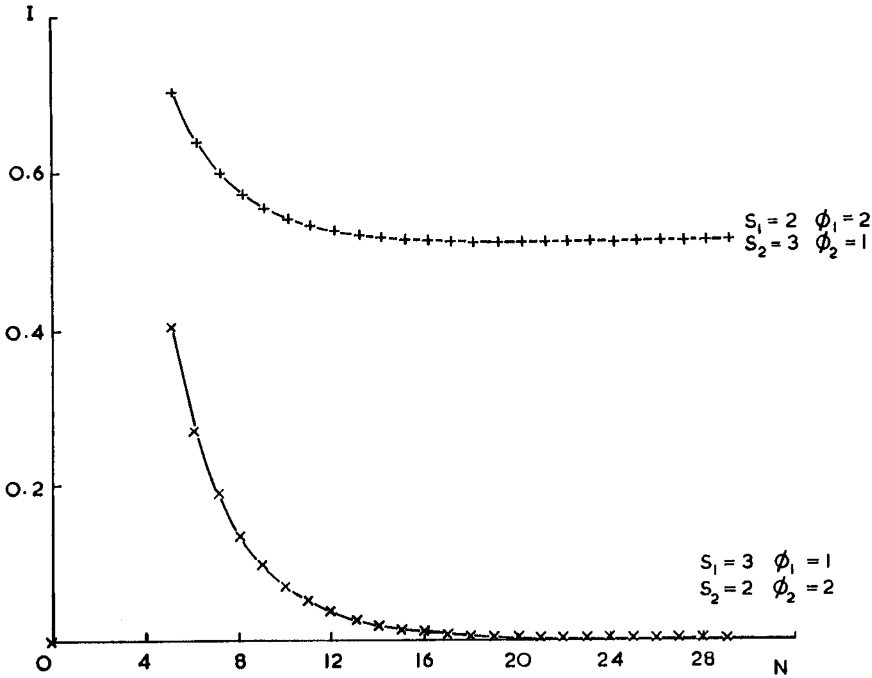

Fig. 2.3 The dependence of the average number of machines idle on the number of spare components

Abbreviating $\pi(n, N-n)$ to $\pi(n)$ we have that, when $N > s_1 + s_2$,

$$\begin{aligned}\pi(n) &= \frac{B_N}{\phi_1^n n! \phi_2^{N-n} s_2! s_2^{N-n-s_2}} && 0 \le n \le s_1 \\ &= \frac{B_N}{\phi_1^n s_1! s_1^{n-s_1} \phi_2^{N-n} s_2! s_2^{N-n-s_2}} && s_1 \le n \le N - s_2 \\ &= \frac{B_N}{\phi_1^n s_1! s_1^{n-s_1} \phi_2^{N-n} (N-n)!} && N - s_2 \le n \le N\end{aligned}$$

Of course n is a birth and death process, and this fact could have been used to derive the above expressions. The normalizing constant B_N is determined by the identity $\sum \pi(n) = 1$, and elementary calculations show that

$$B_N = \frac{s_1! s_1^{-s_1} s_2! s_2^{N-s_2} \phi_2^N}{s_1! s_1^{-s_1} F(\rho s_1, s_1) + (\rho^{s_1+1} - \rho^{N-s_2})/(1-\rho) + s_2! s_2^{N-s_2} (\rho/s_2)^N F(s_2/\rho, s_2)}$$

where

$$\rho = \frac{\phi_2 s_2}{\phi_1 s_1}$$

and

$$F(x, r) = \sum_{n=0}^{r} \frac{x^n}{n!}$$

In equilibrium the average number of machines idle is

$$I = \sum_{n=0}^{s_1} (s_1 - n)\pi(n)$$

The dependence of I on N is illustrated in Fig. 2.3.

A mining operation. Consider a sequence of coal faces which are worked on in turn by a number of specialized machines. Examples of machines might be a cutting machine, a loading machine, and a roofing machine. Each machine proceeds to the next face after completing its task. We could regard the machines as queueing up at faces (Fig. 2.4a). However, the faces will

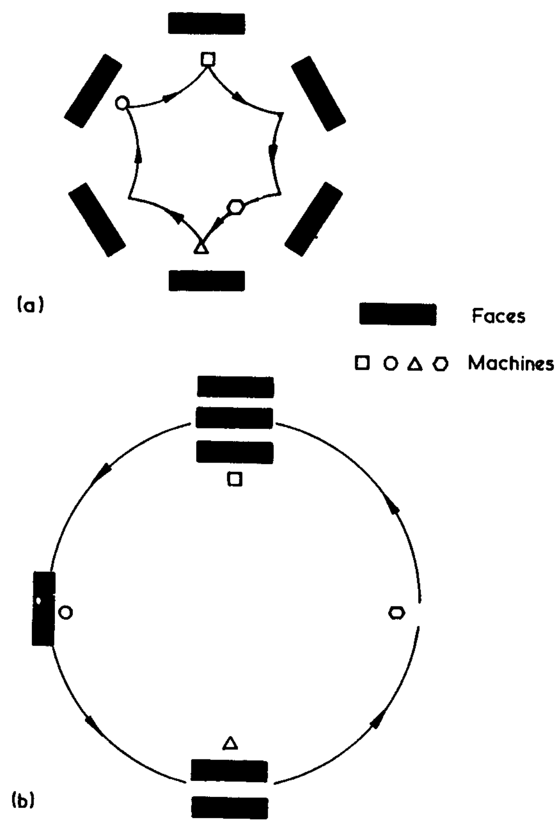

Fig. 2.4 A mining operation

usually be a more homogeneous group than the machines, and for this reason we shall regard the machines as fixed and the faces as queueing up for service from the machines (Fig. 2.4b). Suppose now that there are J machines and N faces and that the time taken by machine j to deal with a face is exponentially distributed with mean ϕ_j^{-1}, for $j=1,2,\ldots,J$. The system will then be a cyclic queue. If n_j is the number of faces queueing at machine j, then in equilibrium

$$\pi(n_1, n_2, \ldots, n_J) = \frac{B_N}{\prod_{j=1}^J \phi_j^{n_j}}$$

Note that the equilibrium probabilities do not depend upon the order in which machines work on faces. The normalizing constant is

$$B_N = \left[\sum_{\mathbf{n}\in\mathcal{S}} \phi_1^{-n_1}\phi_2^{-n_2}\cdots\phi_J^{-n_J}\right]^{-1}$$

and various quantities of interest depend upon it. For example the identity

$$\phi_j^{-1}B_{N-1}^{-1} = \left[\sum_{\mathbf{n}\in\mathcal{S}:n_j>0} \phi_1^{-n_1}\phi_2^{-n_2}\cdots\phi_J^{-n_J}\right]$$

allows the probability that machine j is working to be written as

$$\sum_{\mathbf{n}\in\mathcal{S}:n_j>0} \pi(\mathbf{n}) = \frac{B_N}{\phi_j B_{N-1}}$$

An interesting phenomenon emerges as $N\to\infty$ if one of the machines is slower than the rest. Suppose that $\phi_1<\phi_j$, $j=2,3,\ldots,J$. Then as $N\to\infty$ queue 1 will become a bottleneck with most of the customers in the system waiting there, and the arrival process at queue 2 will become more and more like a Poisson process. In the limit queues $2,3,\ldots,J$ will behave as the series of queues considered in Section 2.2. This point is developed further in Exercise 2.4.5.

Exercises 2.3

1. Show that if the process $\mathbf{n}(t)$ is a closed migration process with transition rates (2.1) then the reversed process $\mathbf{n}(-t)$ is also a closed migration process, with transition rates

$$q'(\mathbf{n}, T_{jk}\mathbf{n}) = \lambda'_{jk}\phi_j(n_j)$$

where

$$\lambda'_{jk} = \frac{\alpha_k\lambda_{kj}}{\alpha_j}$$

Show that in equilibrium the probability flux that an individual moves from colony j to k is

$$\frac{\alpha_j \lambda_{jk} B_N}{B_{N-1}}$$

and deduce that in equilibrium the mean arrival rate at colony j, expression (2.7), is

$$\frac{\alpha_j \lambda_j B_N}{B_{N-1}}$$

2. Figure 2.3 suggests that I tends to a limit as $N \to \infty$. Prove this and show that the limit is zero if $\rho \geq 1$ and is $(1-\rho)s_1$ if $\rho \leq 1$.
3. Suppose that in the model of a mining operation $\phi_1 = \phi_2 = \cdots = \phi_J = \phi$. Show that in equilibrium the probability a given machine is operating is $N/(N+J-1)$ and that the average time for a machine to complete one cycle of faces is $(N+J-1)/\phi$.
4. Show that in the model of a mining operation the mean number of faces queueing for machine j can be written as

$$E(n_j) = \frac{\phi_j}{B_N}\left(\frac{\partial B_N}{\partial \phi_j}\right)$$

5. Consider a closed migration process in which each colony is a single-server queue. Suppose that a capacity constraint is put on each queue by the prohibition of any transition which would raise n_j above R_j, $j = 1, 2, \ldots, J$. Thus if $R = \sum_{j=1}^{J} R_j$ then we must have $R \geq N$. Suppose in addition

$$R - R_j < N \qquad j = 1, 2, \ldots, J$$

so that no queue can become empty. Show that if $m_j = R_j - n_j$ then $(m_1, m_2, \ldots, m_J)$ is a closed migration process, and hence deduce the equilibrium distribution for $(n_1, n_2, \ldots, n_J)$.
6. Show that the number of distinct states in a closed migration process is

$$\binom{N+J-1}{J-1}$$

Thus to calculate B_N directly as a sum of terms is impractical for even relatively small values of N and J. Fortunately there is an alternative. Define the generating functions

$$\Phi_j(z) = \sum_{n=0}^{\infty} \frac{(\alpha_j z)^n}{\prod_{r=1}^{n} \phi_j(r)}$$

$$B(z) = \sum_{N=0}^{\infty} \frac{z^N}{B_N}$$

Show that

$$B(z)=\prod_{j=1}^{J}\Phi_j(z)$$

Thus B_N can be calculated by multiplying together the functions $\Phi_j(z)$, $j=1,2,\ldots,J$, after they have each been truncated to the first $N+1$ terms. The number of steps required to do this is of order JN^2, and so this method is computationally much more efficient.

7. The generating function method readily yields marginal distributions. If β_n is the coefficient of z^N in

$$\frac{(\alpha_k z)^n}{\prod_{r=1}^{n}\phi_k(r)}\prod_{j\neq k}\Phi_j(z)$$

show that the probability colony k contains n individuals is $\beta_n B_N$.

8. In special cases the amount of computation required by the generating function method can be reduced further. If each colony is a single-server queue show that the form of the functions $\Phi_j(z)$, $j=1,2,\ldots,J$, allows B_N to be calculated in order JN steps. Show also that the probability queue k contains n or more customers is

$$\frac{\alpha_k^n B_N}{\phi_k^n B_{N-n}}$$

2.4 OPEN MIGRATION PROCESSES

In this section we shall again consider a set of J colonies but we shall allow individuals to enter and leave the system as well as to move between colonies. We will require the operators $T_{j\cdot}$ and $T_{\cdot k}$ defined as follows:

$$T_{j\cdot}(n_1,n_2,\ldots,n_j,\ldots,n_J)=(n_1,n_2,\ldots,n_j-1,\ldots,n_J)$$

$$T_{\cdot k}(n_1,n_2,\ldots,n_k,\ldots,n_J)=(n_1,n_2,\ldots,n_k+1,\ldots,n_J)$$

Thus $T_{j\cdot}$ removes an individual from colony j and $T_{\cdot k}$ introduces one at colony k. We shall study $\mathbf{n}$ under the assumption that it is a Markov process with transition rates given by

$$\begin{aligned} q(\mathbf{n},T_{jk}\mathbf{n})&=\lambda_{jk}\phi_j(n_j)\\ q(\mathbf{n},T_{j\cdot}\mathbf{n})&=\mu_j\phi_j(n_j)\\ q(\mathbf{n},T_{\cdot k}\mathbf{n})&=\nu_k \end{aligned}\tag{2.8}$$

where $\phi_j(0)=0$. We shall require that $\phi_j(n)>0$ if $n>0$ and that the parameters λ_{jk}, μ_j, and ν_k allow an individual to reach any colony from outside the system and to leave the system from any colony, either directly or indirectly via a chain of other colonies. Under these conditions the process $\mathbf{n}$ is irreducible within the state space $\mathbb{N}^J$, and we shall call it an open migration process.

The major difference between a closed and an open migration process is that in the latter individuals arrive at colonies from outside the system and individuals leaving colonies may well leave the system entirely. The transition rates (2.8) imply that arrivals at colony k from outside the system form a Poisson process of rate ν_k and that when an individual leaves colony j he will leave the system with probability μ_j/λ_j where

$$\lambda_j = \mu_j + \sum_k \lambda_{jk}$$

It is often convenient to scale the function ϕ_j so that $\lambda_j = 1$.

A series of simple queues (Fig. 2.2) is an example of an open migration process with $\phi_j(n) = \phi_j$, $n > 0$, where ϕ_j is the service rate at queue j, and with the only other non-zero parameters being $\nu_1 = \nu$ and $\lambda_{12} = \lambda_{23} = \cdots = \lambda_{J-1,J} = \mu_J = 1$. If we alter this system by setting $\mu_J = \lambda_{J1} = \frac{1}{2}$ we obtain the open migration process illustrated in Fig. 2.5, in which when a customer leaves queue J he returns to queue 1 with probability $\frac{1}{2}$ and leaves the system otherwise.

The conditions we have imposed on the parameters λ_{jk}, μ_j, and ν_k ensure that the equations

$$\alpha_j\left(\mu_j + \sum_k \lambda_{jk}\right) = \nu_j + \sum_k \alpha_k \lambda_{kj} \qquad j = 1, 2, \ldots, J \tag{2.9}$$

have a unique solution for $\alpha_1, \alpha_2, \ldots, \alpha_J$, which is positive (Exercise 2.4.1). We shall require as normalizing constants $b_1, b_2, \ldots, b_J$, where

$$b_j^{-1} = \sum_{n=0}^{\infty} \frac{\alpha_j^n}{\prod_{r=1}^n \phi_j(r)}$$

Let b_j be zero if the sum is infinite.

Theorem 2.4. *If $b_1, b_2, \ldots, b_J$ are all positive then the open migration process has an equilibrium distribution. In equilibrium $n_1, n_2, \ldots, n_J$ are independent and*

$$\pi_j(n_j) = b_j \frac{\alpha_j^{n_j}}{\prod_{r=1}^{n_j} \phi_j(r)} \qquad j = 1, 2, \ldots, J \quad .$$

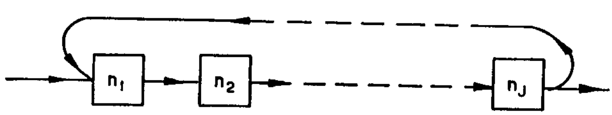

Fig. 2.5 An open migration process

Proof. The equilibrium equations are

$$\pi(\mathbf{n})\left[\sum_j q(\mathbf{n}, T_{j\cdot}\mathbf{n})+\sum_j \sum_k q(\mathbf{n}, T_{jk}\mathbf{n})+\sum_k q(\mathbf{n}, T_{\cdot k}\mathbf{n})\right]$$

$$=\sum_j \pi(T_{j\cdot}\mathbf{n})q(T_{j\cdot}\mathbf{n}, \mathbf{n})+\sum_j \sum_k \pi(T_{jk}\mathbf{n})q(T_{jk}\mathbf{n}, \mathbf{n})+\sum_k \pi(T_{\cdot k}\mathbf{n})q(T_{\cdot k}\mathbf{n}, \mathbf{n})$$

which will be satisfied if we can find a distribution $\pi(\mathbf{n})$ which satisfies the partial balance equations

$$\pi(\mathbf{n})\left[q(\mathbf{n}, T_{j\cdot}\mathbf{n})+\sum_k q(\mathbf{n}, T_{jk}\mathbf{n})\right]$$

$$= \pi(T_{j\cdot}\mathbf{n})q(T_{j\cdot}\mathbf{n}, \mathbf{n})+\sum_k \pi(T_{jk}\mathbf{n})q(T_{jk}\mathbf{n}, \mathbf{n}) \qquad j=1,2,\ldots,J$$

and

$$\pi(\mathbf{n}) \sum_k q(\mathbf{n}, T_{\cdot k}\mathbf{n}) = \sum_k \pi(T_{\cdot k}\mathbf{n})q(T_{\cdot k}\mathbf{n}, \mathbf{n})$$

Substitution will verify that

$$\pi(\mathbf{n}) = B\prod_{j=1}^{J} \frac{\alpha_j^{n_j}}{\prod_{r=1}^{n_j} \phi_j(r)} \tag{2.10}$$

satisfies the partial balance equations. For example the final partial balance equation reduces to, after substitution,

$$\sum_k \nu_k = \sum_k \alpha_k \mu_k$$

the truth of which is established by summing equations (2.9). Since $b_1, b_2, \ldots, b_J$ are positive the choice $B = b_1 b_2 \cdots b_J$ ensures that $\pi(\mathbf{n})$ sums to unity. Thus $\pi(\mathbf{n})$ is the equilibrium distribution and the independence of $n_1, n_2, \ldots, n_J$ follows from the fact that both $\pi(\mathbf{n})$ and the state space $\mathbb{N}^J$ have a product form.

The independence established in Theorem 2.4 is of the random variables $n_1, n_2, \ldots, n_J$ observed at a fixed point in time. Viewed as stochastic processes, defined for $t \in \mathbb{R}$, $n_1(t), n_2(t), \ldots, n_J(t)$ are clearly not independent.

It is interesting to note that the equilibrium distribution for colony j is just what it would be if it were the only colony in the system, with individuals arriving there in a Poisson stream of rate $\alpha_j\lambda_j$ and leaving at rate $\lambda_j\phi_j(n_j)$. This is especially intriguing since the combined arrival process at a colony, from other colonies and from outside, is not in general Poisson (Exercise 2.4.2).

If any of $b_1, b_2, \ldots, b_J$ are zero the process does not have an equilibrium distribution: there is a colony which individuals enter more quickly than they leave.

Observe that for the process to be reversible $\alpha_1, \alpha_2, \ldots, \alpha_J$ must satisfy

$$\begin{aligned} \alpha_j\lambda_{jk} &= \alpha_k\lambda_{kj} \\ \alpha_j\mu_j &= \nu_j \end{aligned} \tag{2.11}$$

Even when the process is not reversible the reversed process is of a similar form.

Theorem 2.5. *If* $\mathbf{n}(t)$ *is a stationary open migration process then so is the reversed process* $\mathbf{n}(-t)$.

Proof. Using Theorem 1.12 the transition rates of the reversed process $\mathbf{n}(-t)$ can be calculated from the rates (2.8) and the equilibrium distribution (2.10). For example

$$\begin{aligned} q'(\mathbf{n}, T_{jk}\mathbf{n}) &= \frac{\pi(T_{jk}\mathbf{n})q(T_{jk}\mathbf{n}, \mathbf{n})}{\pi(\mathbf{n})} \\ &= \lambda'_{jk}\phi_j(n_j) \end{aligned}$$

where

$$\lambda'_{jk} = \frac{\alpha_k\lambda_{kj}}{\alpha_j}$$

Similarly,

$$q'(\mathbf{n}, T_{j\cdot}\mathbf{n}) = \mu'_j\phi_j(n_j)$$

and

$$q'(\mathbf{n}, T_{\cdot k}\mathbf{n}) = \nu'_k$$

where

$$\mu'_j = \frac{\nu_j}{\alpha_j}$$

and

$$\nu'_k = \alpha_k\mu_k$$

Thus the reversed process is also an open migration process.

Call the points in time at which an individual leaves the system from colony j the exit process from colony j. By the departure process from colony j we shall mean the points in time at which an individual leaves colony j, either for another colony or to leave the system.

Corollary 2.6. *If $\mathbf{n}(t)$ is a stationary open migration process then the exit process from colony j is a Poisson process of rate $\alpha_j\mu_j$. Further, the exit processes from colonies $1, 2, \ldots, J$ are independent and $\mathbf{n}(t_0)$ is independent of the exit processes prior to time t_0.*

Proof. In the reversed process arrivals at colony j from outside the system form a Poisson process of rate $\nu_j' = \alpha_j\mu_j$. But these arrivals correspond precisely to departures from the system in the original process, and the result follows.

Neither the departure process from a colony nor the stream of customers moving from one colony to another is in general Poisson (Exercise 2.4.2). This again is intriguing. In the migration process illustrated in Fig. 2.5 an individual leaving colony J chooses at random and independently of everything that has gone before whether to leave the system or to return to colony 1. Yet while the departure process from colony J is not Poisson the exit process is. Note that the individual's decision on whether or not to return to colony 1 may be independent of past departures, but it is not independent of future departures.

Corollary 2.7. *Suppose that colony j in a stationary open migration process is a queue with s servers at which the queue discipline is first come first served. Let $\phi_j(n) = \phi_j \min(n, s)$ and $\lambda_j = 1$, so that service times are exponentially distributed with mean ϕ_j^{-1}. Then the waiting time of a customer at queue j has the same distribution as if queue j were an isolated $M/M/s$ queue with a Poisson arrival process of rate α_j.*

Proof. In a stationary open migration process the probability flux that a customer departs from queue j leaving n_j customers behind is $\pi_j(n_j + 1)\lambda_j\phi_j(n_j + 1)$. Thus if at time t a customer leaves queue j the probability there will be n_j customers left in queue j is

$$\frac{\pi_j(n_j+1)\phi_j(n_j+1)}{\sum_{r=0}^{\infty} \pi_j(r+1)\phi_j(r+1)} = \pi_j(n_j)$$

Consideration of the reversed process $\mathbf{n}(-t)$ shows that this is also the probability that a typical customer arriving at queue j finds n_j customers already there. But $\pi_j(n_j)$ is just what this probability would be if queue j were in isolation with customers arriving in a Poisson stream of rate α_j. The queue discipline ensures that the distribution of the waiting time of a customer is determined by the distribution of the number of customers he finds on his arrival, and the result follows.

Some of the simplest examples of open migration processes are those for which $\phi_j(n) = n$ for all j, i.e. linear migration processes. For these,

$b_1, b_2, \ldots, b_J$ will always be positive and so an equilibrium distribution will always exist. Indeed,

$$\pi_j(n_j) = e^{-\alpha_j} \frac{\alpha_j^{n_j}}{n_j!}$$

so that the number of individuals in colony j has a Poisson distribution. This result provides an alternative interpretation for the constants $\alpha_1, \alpha_2, \ldots, \alpha_J$; α_j is the expected number of individuals in colony j when individuals move independently with transition intensities λ_{jk}, μ_j, and ν_k.

Until now we have assumed that the number of colonies, J, is finite. In fact the proof of Theorem 2.4 goes through unchanged when J is infinite provided $B = b_1 b_2 \cdots$ is positive; note that when this is so the equilibrium distribution (2.10) assigns probability one to the countable set of states satisfying $\sum n_j < \infty$. In the following example we discuss a linear migration process with J infinite.

The family size process. Consider the following elaboration of the simple birth, death, and immigration process described in Section 1.3. Suppose that each immigrating individual has a distinguishing characteristic, such as a genetic type or a surname, which is passed on to all his descendants but which is not shared by any other individual. Thus at any point in time the population can be divided into distinct families, each of which consists of all those individuals alive with a given characteristic. Let n_j be the number of families of size j. Then the family size process $(n_1, n_2, \ldots)$ is a linear migration process with transition rates

$$\begin{aligned} q(\mathbf{n}, T_{j,j+1}\mathbf{n}) &= j\lambda n_j \qquad j = 1, 2, \ldots \\ q(\mathbf{n}, T_{j,j-1}\mathbf{n}) &= j\mu n_j \qquad j = 2, 3, \ldots \\ q(\mathbf{n}, T_{\cdot 1}\mathbf{n}) &= \nu \\ q(\mathbf{n}, T_{1\cdot}\mathbf{n}) &= \mu n_1 \end{aligned}$$

Observe that a *family* is the basic unit which moves through the colonies of the system and that the movements of different families are independent. Equations (2.9) have the solution

$$\alpha_j = \frac{\nu}{\lambda j} \left(\frac{\lambda}{\mu}\right)^j$$

and the normalizing constant $B = \exp(-\sum \alpha_j)$ is positive provided $\lambda < \mu$, since then $\sum \alpha_j$ is finite. In equilibrium the process is reversible, the number of families of size j has a Poisson distribution with mean α_j, and the total number of families in the system has a Poisson distribution with mean

$$\sum \alpha_j = -\frac{\nu}{\lambda} \log\left(1 - \frac{\lambda}{\mu}\right)$$

Optimal allocation of effort. In this example we shall discuss an optimization application of Theorem 2.4. Consider an open migration process in which each colony is a single-server queue: suppose $\lambda_j = 1$, $\phi_j(n) = \phi_j$, $n > 0$, for $j = 1, 2, \ldots, J$. For equilibrium at each queue the service rate (or *effort*) ϕ_j must be greater than the mean arrival rate (or *demand*) α_j, and then

$$\pi_j(n_j) = \left(1 - \frac{\alpha_j}{\phi_j}\right)\left(\frac{\alpha_j}{\phi_j}\right)^{n_j}$$

Thus the mean number of customers present at queue j is $\alpha_j/(\phi_j - \alpha_j)$. Suppose now that we have control over the values of $\phi_1, \phi_2, \ldots, \phi_J$, subject to the constraint

$$\sum_j \phi_j = F$$

How should we choose $\phi_1, \phi_2, \ldots, \phi_J$ to minimize the mean number of customers present in the system? This problem can be readily solved using Lagrangian multipliers. Let

$$L = \sum_j \frac{\alpha_j}{\phi_j - \alpha_j} + y\left(\sum_j \phi_j - F\right)$$

Setting $\partial L/\partial \phi_j = 0$ we find that L is minimized by the choice

$$\phi_j = \alpha_j + \sqrt{\frac{\alpha_j}{y}}$$

Substituting this into the constraint shows that we should choose

$$\frac{1}{\sqrt{y}} = \frac{F - \sum_k \alpha_k}{\sum_k \sqrt{\alpha_k}}$$

Hence the optimal allocation is

$$\phi_j = \alpha_j + \frac{\sqrt{\alpha_j}}{\sum_k \sqrt{\alpha_k}}\left(F - \sum_k \alpha_k\right) \qquad j = 1, 2, \ldots, J$$

Thus the optimal allocation proceeds by first giving to each queue just enough to satisfy demand and then by allocating the surplus, $F - \sum_k \alpha_k$, in proportion to the square roots of the demands. This result is mildly surprising; we might have thought that effort would be allocated in proportion to demand. Relative to this allocation the optimal allocation concentrates less effort on those queues with high demands.

Little's result (1.12) shows that the optimal allocation also minimizes the mean period spent in the system by a customer.

A further discussion is contained in Section 4.1.

Exercises 2.4

1. By considering a Markov process with $J+1$ states and transition rates

$$q(j,k)=\lambda_{jk} \qquad j,k=1,2,\ldots,J$$
$$q(j,0)=\mu_j \qquad j=1,2,\ldots,J$$
$$q(0,k)=\nu_k \qquad k=1,2,\ldots,J$$

show that equations (2.9) have a unique solution and that this solution is positive. Show that $\alpha_j\lambda_j/\sum \nu_k$ is the expected number of times the jump chain of this process visits state j between successive visits to state 0. Deduce that in an open migration process $\alpha_j\lambda_j$ is the mean arrival rate at colony j, counting arrivals from outside the system and from other colonies. Obtain the same result by calculating the probability flux that an individual leaves colony j.

2. Consider the open migration process illustrated in Fig. 2.5 with $J=2$, $\phi_j(n)=n$, $j=1,2$, $\mu_1=\nu_2=0$, $\lambda_{12}=\lambda_{21}=\lambda$, $\nu_1=\nu$, and $\mu_2=\mu$. Show that the arrival process at colony 1, counting arrivals from outside the system and from colony 2, comprises a Poisson process of rate ν together with for each point of this process a string of further points, where the number of further points in each string is geometrically distributed with mean λ/μ and the interval between points in the same string has mean $2\lambda^{-1}$. Suppose now that ν is small and λ large. Show that the arrival process at colony 1 is not Poisson. Deduce that the departure process from colony 2 is not Poisson. Show that the points in time at which individuals move from colony 2 to colony 1 do not form a Poisson process.

3. Consider an open migration process. If it is not possible for an individual in colony k ever to reach colony j show that the stream of individuals moving from colony j to colony k is Poisson.

4. Consider an open migration process in which an individual can never visit a colony more than once, and the graph G, with an edge joining nodes j and k if either λ_{jk} or λ_{kj} is positive, is a tree. If each queue is a first come first served single-server queue show that the waiting times of a customer at the queues he visits are independent. Note that the conditions ensure that customers cannot overtake one another. Using Exercise 2.2.4 show that the conclusion remains valid if some queues have more than one server provided these queues are such that a customer can only visit them immediately on entering or immediately prior to leaving the system.

5. Consider a stationary closed migration process with colonies labelled $0,1,2,\ldots,J$ and with $\lambda_{j0}=\mu_j$, $\lambda_{0j}=\nu_j$, for $j=1,2,\ldots,J$, and $\phi_0(n)=1$, $n>0$. Let $\alpha_1,\alpha_2,\ldots,\alpha_J$ be the solution to equations (2.9) and suppose the constants $b_1,b_2,\ldots,b_J$ calculated from this solution are all positive.

Show that if the number of individuals in the system is increased towards infinity the behaviour of colonies $1, 2, \ldots, J$ approaches that of an open migration process.

6. The equilibrium distribution obtained for the family size process should be consistent with that found for the population size in Section 1.3. Establish this directly by showing that if

$$M = \sum_{j=1}^{\infty} j n_j$$

where $n_1, n_2, \ldots$ are independent random variables, n_j Poisson with mean

$$\frac{\nu}{\lambda j}\left(\frac{\lambda}{\mu}\right)^j$$

then M has the negative binomial distribution (1.14).

7. In the family size process show that the total number of individuals M and the total number of families

$$N = \sum_{j=1}^{\infty} n_j$$

satisfy the relations

$$E(N) = \frac{\nu}{\lambda} \log\left(1 + \frac{\lambda}{\nu} E(M)\right)$$

and

$$\operatorname{cov}(M, N) = \frac{\nu}{\mu - \lambda}$$

8. Consider the family size process. Show that if an individual is the only member of his family then he is an immigrant who has not yet given birth with probability $\mu/(\mu + \lambda)$.
9. In the family size process show that the points in time at which a family becomes extinct form a Poisson process. Show that this remains true even when the model is amended to allow the birth of twins.